Cheg 355 Transport I

This semester we will study Fluid Mechanics: the motion of fluids (and solids) in response to applied forces such as shear or pressure, or body forces ranging from gravity to electro kinetic or magnetic forces.

We will use conservation principles to derive mathematical description of simple & complex phenomena. Such mathematical models can be used to understand and predict phenomena, and solve problems in engineering.

The first HW is already due - it's just a few practice problems to review vector calculus.

Texts:
1) Bird, Stewart, & Lightfoot, Transport Phenomena - the updated version of the class text.
   This should be available in the bookstore soon, and is a useful ref.
2) The course notes - we're still figuring out the best way of distributing these due to the new copyright regulations. Printed copies will be available soon, but online versions are up now!

Check the online version periodically, as the notes may be updated during the semester.

Admin Details:
- Weekly HW (15%)
- 2-hour exams (25% each)
- Final exam (35%)
We'll also have a weekly tutorial:
Mondays 6:00-8:00 PM, (likely)
(note: tutorials are optional)
The first tutorial will be a discussion of index notation.

TA's:
- Garrett Tow
- Sihan Yu

The notes, HW, etc will be posted to the website:
www.mk.ece/cle/hr/cle355/cle355.html

OK, why should we care about fluids??

⇒ Vital to the world around us!
  - What causes a hurricane & determines its path? A tornado?
  - How do you design a sprinkler system so that all areas are doused equally in case of fire?
  - How can you design an artificial heart so that it pumps blood without tearing up blood cells?
  - How can you mix fluids in a chip-based HTS system
All these questions are answered by applying fundamental conservation laws as well as material properties to complex systems!

What is conserved?

- Mass (neither created nor destroyed)
- Momentum ($F = ma$)
- Energy (we'll get there eventually...)

We will apply these conservation laws to fluids, but they apply equally well to solids (or anything in between!)!

What is a fluid? Fluids exhibit continuous deformations that do not snap back after stress is removed (thermodynamics: state of fluid depends on rate of shear).

Solids: Elastic deformation - like a rubber band, snaps back after stress is removed (thermodynamics: state depends on total deformation). Virtually everything lies between these two states!

Examples: metal creep, elastic polymer flow.

Properties of Fluids

If we characterize fluids by rate of deformation, most important prop. relates to resistance to deformation ⇒ viscosity!

We have a thought experiment: put mat'1 in a gap between plates: $F \rightarrow \Delta x$.

If mat'1 is elastic solid, we get some fixed displacement $\Delta x$ for a given force $F$ at SS.

If linearly elastic, relation $\frac{E}{A} = \frac{\Delta x}{D}$.

$E$ = Young's modulus of elasticity.

What are units of $E$? ⇒ same as $F/A$! Usually given as psi, dynes/cm², etc.!

What units to use?? - Depends on application, but you should know all of them! ⇒ know how to convert!

I usually use cgs - most appropriate for low Re flow (stability). McCrady would use mks - high Re, oil systems in English units ⇒ all are the same physics!
Q. We fill it with a fluid. What happens? => will get continuous deformation! Plate will move up some velocity \( \frac{dU}{dt} \).

For a Newtonian fluid \( \frac{U}{D} \rightarrow g \) (poise)

"poise" is short for Poiseuille; name assoc. w/ pipe flow.

\( \dot{\gamma} \) is rate of strain => known as shear rate.

Velocity field is known as plane Couette flow, simple shear flow.

You should get to know the jargon:

- What are the viscosities of some simple fluids?
- Water \( \approx 1 \) cp (centipoise, \( 10^{-2} \) poise)
- Hair Spray \( \approx 30 \) p (temp. dep.)
- Air \( \approx 0.02 \) cp

All these are Newtonian fluids!

What are ex. of non-Newtonian fluids?

- One feature is stress-strain relation is not linear (or may not be).

\[ \dot{\gamma} \]

\( \dot{\gamma} \) => shear rate

1. Bingham plastic \( \rightarrow \) a linear relation betw. \( \tau \) & \( \dot{\gamma} \), but there is a yield stress \( \Rightarrow \) no motion until critical strain exceeded! Ex: frozen OJ, mayo

2. Dilatant \( \Rightarrow \mu \) increases w/ \( \dot{\gamma} \)

Not seen as often - some clay suspensions do this

3. Newtonian

4. Pseudoplastic \( \Rightarrow \mu \) decreases w/ \( \dot{\gamma} \)

Also called shear thinning - very common in polymer melts!

\( \mu \) may be much more complicated than this! \( \mu \) may be time dep., may exhibit combination of phenomena.

- Example: liquid chocolate - exhibits yield stress & shear thinning! Imp. if fabricating chocolate figures!

Other examples: cytological fluid:

\[ \dot{\gamma} \]

Indeterminate shear rate for applied shear stress! Leads to complex patterns in cytological streaming!

Normal stresses \( \tau \) may not be a scalar! \( \Rightarrow \) if you shear fluid one way, may get stress in a different direction! Arises in fluids w/ structure.
Other properties:
- Speed of Sound \( V_s \) — important in jet aircraft, high speed machinery.

Related to compressibility of fluid:
- Sound is a pressure wave travelling thru a fluid \( V_s = \frac{\gamma}{\gamma-1} \sqrt{\frac{RT}{\rho}} \).

For an ideal gas \( P = \frac{\gamma}{M} RT \).
Thus \( \left( \frac{\gamma P}{\gamma-1} \right)_T = \left( \frac{\gamma}{M} \right) (\gamma-1) (273 + 298) \text{ (29.9/4.4}) \)
\[ = 8.6 \times 10^5 \text{ cm}^2/\text{s} \]
Thus \( V_s = 2.9 \times 10^4 \text{ cm/s} \)
\[ = 655 \text{ mph} \]

Result is a “tension” along the surface \( \Delta P \) + higher pressure within concave side of bubble.
- \( \Delta P \propto \frac{\rho}{R} \) (inverse to radius).

Surfactants (soap) are a material that likes both fluids, thus reduces \( \sigma \).

Coefficient of thermal expansion:
\[ \beta = -\frac{1}{3} \left( \frac{\partial^2 \rho}{\partial T^2} \right) \rho + \left( \frac{\partial V}{\partial T} \right)_{\rho} = \frac{1}{\gamma} \]
for an ideal gas.

Important in natural convection problems, such as draft off window — will look at this in Sr. Lab.

When \( \gamma V_s \ll 1 \) flow is compressible
\( \Rightarrow \) this means that fluid density is affected by fluid motion.
Importance gauged by Mach \( M = \frac{\gamma V}{V_s} \).

For liquids \( \left( \frac{\gamma P}{\gamma-1} \right)_T \) is very large & \( \gamma \) is usually smaller, so flow can be regarded as incompressible

Surface Tension: usually denoted by \( \sigma \) (sometimes \( \gamma \)).
\( \sigma \Rightarrow \) energy required to create interfacial surface area.

units = \( \text{g cm}^{-2} \)
This causes bubbles to be spheres! (minimize surface/volume)

Ok, what types of flows are there?
- Compressible vs. Incompressible — depends on \( M = \frac{\gamma V}{V_s} \)
- Even in air, most flows are incompressible! Usually study compressible flows in Aero.

Laminar vs. Turbulent
- Flow is laminar if layers of fluid slip smoothly over each other.
- Laminar flow may be steady (unchanging in time) or unsteady
\( \Rightarrow \) looks at flow from top. At low flows, looks like a glossy, steady stream.
Suspensions ⇒ area of research at
NB. Example - wet sand - if
you step on it, it dries out!
Study of stress-strain relationship is rheology

2nd property: Density
⇒ we are interested in transport
of momentum which is velocity × mass
⇒ density is important!

Density of Water = 1.2 g/cm³
air = 1.2 × 10⁻³ g/cm³
Hg = 13.6 g/cm³

Actually, we are interested in

What do these numbers mean?
Determine time to approach steady-state!
Thought exp't ⇒ take metal
pole, stick one end in fire—
eventually, your hand gets sick!
How long? Controlled by diffusion
Remember: [x] = \frac{L^2}{T}

Thus \( T \sim \frac{L^2}{D} \)

for a metal, \( L \approx 0.11 \text{ cm}^2 / \text{s} \) (steel)
Thus if pole is 2 ft long (60 cm)
\[ \text{it takes} \ O(10) \text{ hr for your end to get hot!} \]

Actually, more complicated,
as losses heat to air all along shaft

\[ \text{momentum diffusivity} \Rightarrow \ \frac{L^2}{D} \]
(better known as kinematic viscosity
\[ \nu \Rightarrow \frac{L^2}{D} \]
units of \( \nu \) same as molecular diff. \( \nu \) governs rate by which mom.
diffuses

<table>
<thead>
<tr>
<th>Material</th>
<th>( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td>1 cs</td>
</tr>
<tr>
<td>Air</td>
<td>15 cs</td>
</tr>
<tr>
<td>Hg</td>
<td>0.5 cs</td>
</tr>
<tr>
<td>Karo Syrup</td>
<td>25 cs</td>
</tr>
</tbody>
</table>

What about fluids? Look at diff^2 of momentum ⇒ same
thought exp't:

How long till lower plate feels motion?

\[ T \sim \frac{h^2}{D} \]

If \( h = 1 \text{ cm} \)

\[ T = O(100 \text{ s}) \text{ in water} \]
\[ = O(200 \text{ s}) \text{ in Hg} \]
\[ = O(0.04 \text{ s}) \text{ in Karo syrup!} \]

Actually, this is only order of magnitude
⇒ soln of transient problem shows ~4x faster than these values.
What happens if we increase flow rate? It becomes rough and variable. Turbulence is chaotic, time dependent, and very difficult to describe mathematically with precision—still it occurs most of the time! Both laminar and turbulent flow may occur in the same geometry. Famous example: pipe flow by Osborne Reynolds. Found transition from laminar to turbulent flow given by dimensionless parameter \( Re \):

\[ Re = \frac{U D}{\nu} \approx 2100 \]

We'll look at this in detail later.

**Continuum Hypothesis**

We want to develop a mathematical description of fluid flow; this requires taking fluid to be a continuum. Is this continuum hypothesis reasonable? Sometimes!

\( \Rightarrow \) Fluid is made up of molecules bouncing into each other. In a gas phase, molecules may go sig. dist. before hitting each other! Not all continuum on this length scale!

Support we have probe of eddies—what would it see? With microscopic variations in local density and microscopic variations in local velocity, pressure, temp., etc. This may not work...! Minimum length for continuum hypothesis to hold is mean-free path length—distance molecule travels before hitting another. In a gas, \( \lambda \sim \frac{1}{\sqrt{n}} \) where \( n \) is number density (molecules/vol).

At 1 atm & room temp, we have \( \lambda \) is just a few μm. For liquids, it's even smaller!

Non-continuum effects are imp. even in liquids, though to the most imp. ex. is Brownian motion. In a liquid, small particles are kicked around by molecules, thus they execute a random walk—gives rise to diffusion—usually imp. for particles 1 μm or less in dia.

We will assume continuum hypothesis to apply, also leads to no-slip condition \( \Rightarrow \) at a solid surface in contact with fluid, velocity is continuous!
Fluid layer adjacent to solid surface moves w/ velocity 

If \( \lambda > D \) (char. length of flow), may not be in contact, so would get a "slip" condition - modifies aerodynamics of re-entry, etc. in a vacuum pump. Also get breakdown of continuum hyp. in composite media, etc. (sup) - not valid on length scales of order particle size. 

leads to wall slip as well, makes working with suspensions tricky. 

When we will describe motion \( \sigma, \mu, \rho, \varepsilon \), force, etc. at a "point" really mean some avg over a volume large with \( \lambda \) or molecule (particle) size.

**Examples:**

- Gravity: \( F = mg \ dv \)  
  - force on a differential volume!

- Electric Field: \( E = \frac{F}{Q} \ dv \)  
  - \( F = qE \) charge \( Q \) electric field \( E \) (vol/cm)  
  - this force is critical in electrophoresis & electrophoresis. We use this effect to separate proteins in our laboratory. 

- Magnetic Field: \( F = J \times B \) magnetic field  
  - current  
  - Important: in plasma dynamics, field of MHD

**Forces on a Fluid Element**

We need to apply \( F = ma \) to a fluid - what are the forces? 

Consider an arbitrary element:

\[ D \] (surface)
\[ D \] (unit normal)

What are the forces on the molecules in \( D \)? Divide into surface forces & body forces. 

What is a body force? \( \Rightarrow \) They act on each molecule in \( D \). 

Ok, what about surface forces? 

We divide these into shear forces and normal forces. 

\( \Rightarrow \) surface forces act on the surface of \( D \) 

\( \Rightarrow \) shear forces act tangential to \( D \). The \( F/A \) in simple shear flow is a shear force! 

\( \Rightarrow \) normal forces act normal to the surface. 

Let the \( F/A \) of surface force be \( F = \hat{F} \) - a vector, we resolve into tangential & normal components:

\[ \hat{F} = \frac{\hat{F}}{2 \pi} \] (patch of surface)
If the unit normal to a patch of surface $dA$ is $n$.
Then $F_n = (F \cdot n) n$

We'll look at $F_x$ later, now focus on normal forces!
\( \Rightarrow \) Consider an element at rest.
If it's at rest, shear forces should be zero. Just have normal forces:

\[
\begin{align*}
\sum F_x &= \text{AF}_x - \text{AF}_y \\
\sum F_y &= \text{AF}_y - \text{AF}_z
\end{align*}
\]

Now $\sin \theta = \frac{\text{AF}_y}{\text{AF}_z}$

Thus $\Delta F_x = \Delta F_y \frac{\text{AF}_y}{\text{AF}_z} = 0$

or dividing by $\Delta x \Delta y$:

\[
\frac{\Delta F_x}{\Delta x \Delta y} = \frac{\Delta F_y}{\Delta y \Delta z}
\]

Define $\Delta F_y = -\frac{\sigma_{xx}}{\Delta y}$ (normal stress)

Similarly:

\[
\frac{\Delta F_x}{\Delta x \Delta z} = -\sigma_{zz}
\]

These are normal stresses.
They rep. diagonal elements of the stress tensor!

Stress tensor is momentum flux $T_{ij}$ = force/area exerted by fluid of greater $i$ on fluid of lesser $i$ in $j$ direction.

\[
\sigma_{xx} = \sum_{x=1}^{\# \text{ greater } x \text{ fluid}} \frac{\text{ greater } x \text{ fluid}}{x} - \sum_{x=1}^{\# \text{ lesser } x \text{ fluid}} \frac{\text{ lesser } x \text{ fluid}}{x}
\]

Thus $\sigma_{xx}$ is negative in compression.

Let's do a force balance.
\( \Rightarrow \) Since element is at rest, the net force in each direction must be zero.

The force balance in the x-direction:

\[
\sum F_x = AF_x - AF_y \sin \theta = 0
\]

Component of $AF_y$ in x-dir

Now $\sin \theta = \frac{AF_y}{AF_z}$

Thus $\Delta F_x = \Delta F_y \frac{AF_y}{AF_z} = 0$

or dividing by $\Delta x \Delta y$:

\[
\frac{\Delta F_x}{\Delta x \Delta y} = \frac{\Delta F_y}{\Delta y \Delta z}
\]

Define $\Delta F_x = \frac{-\sigma_{xx}}{\Delta y}$ (normal stress)

Note: B & L defines this backwards (ch 2) \( \Rightarrow \) doesn't change the physics, just the sign!

We'll use the conventional (most common, anyway) definition in this class.

OK, now look at y-direction:

\[
\sum F_y = \Delta F_y - \Delta F_z \cos \theta = \frac{\text{ weight of fluid of }}{\Delta x \Delta y \Delta z}
\]

Recall $\cos \theta = \frac{AF_x}{AF_z}$

Thus (dividing thru):

\[
\frac{\Delta F_y}{\Delta x \Delta z} - \frac{\Delta F_x}{\Delta y \Delta z} = \frac{\text{ weight of fluid of }}{\Delta x \Delta y \Delta z}
\]

Thus $\sigma_{yy}$ vanishes as $\Delta y \to 0!$
Thus
\[ \sigma_{xx} = \sigma_{yy} = \sigma_{zz} \]
* In a fluid at rest, normal stress is isotropic: same in all directions. This normal stress is just \( \sigma = \) no sign!
\[ \sigma = -\sigma_{xx} = \sigma_{yy} = \sigma_{zz} \]
When not at rest, normal stress is, in general, \( \sigma \) not isotropic!
We define
\[ \sigma = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \]
average of the normal stress!
equiv: \( \sigma = -\frac{1}{3}\text{tr}(\sigma) \) \Rightarrow \text{trace}

Or, in limit \( \Delta x \to 0 \):
\[ \frac{-\sigma_{xx}}{\Delta x} + 8\sigma_{x} = 0 \]
Similarly, \( \frac{\sigma_{yy}}{\Delta y} = 8\sigma_{y} \); \( \frac{\sigma_{zz}}{\Delta z} = 8\sigma_{z} \)
which yield \( \sigma = \text{eg}'s \)!

In vector form:
\[ \nabla \cdot \sigma = 8 \boldsymbol{g} \]

Last term was done using shell balances. If you're good at vector notation, there's an easier (better) way!
Consider arbitrary fluid element:
\[ \delta \Omega \]

How does \( \sigma \) vary in a fluid at rest?\( \sigma \)

Look at a fluid element:
\[ \begin{align*}
\text{Normal force at } y &= y \\
\sigma_{yy} &= \frac{\sigma_{yy}}{\text{normal force at } y} \\
\begin{cases}
\sigma_{yy} = \frac{\partial \sigma_{yy}}{\partial y} \\
\end{cases}
\end{align*} \]

Let's do a force balance in the \( x \)-dir:
\[ \varepsilon \frac{\partial \sigma_{xx}}{\partial x} - \sigma_{xx} + 8\sigma_{x} = 0 \]
Divide through:
\[ -\frac{\partial \sigma_{xx}}{\partial x} + 8\sigma_{x} = 0 \]

What are the forces acting on it?

Surface Force:
\[ \int_{\partial \Omega} -\mathbf{F} \cdot d\mathbf{A} \]

Body Force:
\[ \int_{\partial \Omega} \mathbf{F} \cdot d\mathbf{A} \]

So:
\[ \sum \mathbf{F} = 0 \]

Thus:
\[ \int_{\partial \Omega} -\mathbf{F} \cdot d\mathbf{A} + \int_{\partial \Omega} \mathbf{F} \cdot d\mathbf{A} = 0 \]

We now use the Divergence Theorem:
\[ \int_{\partial \Omega} \mathbf{F} \cdot d\mathbf{A} = \int_{\Omega} \nabla \cdot \mathbf{F} \ dv \]

converts surface int. to vol. int!
So: \( \sum b \mathbf{F} = 0 \), \( \sum b \mathbf{V} = 0 \)

Now since \( D \) was completely arbitrary, it must be true at every point in fluid!

Thus \( \nabla \mathbf{P} - \mathbf{g} = 0 \)

or \( \nabla \mathbf{P} = \mathbf{g} \)

It will be a lot easier to derive things this way when we get to fluids in motion!

\[ \text{Let's integrate!} \]

\[ P = \rho g z + \text{const} \]

\[ P \big| \, z = h = P_0 \]

Thus \( \mathbf{P} = P_0 + \rho \mathbf{g} (h - z) \)

\[ \mathbf{P} \text{ is just as true in an open body of water (driving):} \]

\[ \text{How deep do you have to go to reach 1 atm gauge (e.g., above the atmospheric pressure)?} \]

\[ \int \mathbf{g} \, dz \]

\[ \mathbf{P} = \mathbf{P}_0 + 0 \mathbf{g} \cdot \mathbf{h} \]

\[ \mathbf{P} = \mathbf{P}_0 + 1 \text{ atm} \]

\[ \text{If the } \mathbf{P} \text{ across the membrane exceeds the osmotic pressure, water will flow through the membrane!} \]

\[ \text{How deep must the pipe be to (a) get water into the pipe.} \]

\[ \text{(b) get the lighter fresh water all the way to the surface?} \]

1. \( g \mathbf{g}_i h_i = \Delta P_{\text{osm}} \)

\[ g \mathbf{g}_i = 1.04 \, \text{kg/m}^3, \quad \Delta P_{\text{osm}} = 28 \text{cm} \]

\[ \therefore h_i = 275 \text{ m} \]

2. \( g \mathbf{g}_{\text{H}_2\text{O}} h_2 - g \mathbf{g}_s h_2 = \Delta P_{\text{osm}} \)

\[ h_2 = \frac{\Delta P_{\text{osm}}}{g(\mathbf{g}_s - \mathbf{g}_{\text{H}_2\text{O}})} = 7 \text{ km} \]
Another example: Buoyancy
What is the force exerted by fluid on a submerged object?
\[
\vec{F} = -\int_{V_{D}} \rho \vec{g} dV
\]
The pressure distribution in the fluid is the same as if the object were absent if it is at rest! So:
\[
\vec{F} = -\int_{V_{D}} \rho \vec{g} dV = -\rho g \vec{V}
\]
So fluid exerts a force equal to the weight of displaced volume.
(Archimedes, 3rd cent. B.C.)

Fluids in Motion
Now that we’ve dealt with hydrostatics, let’s look at fluids in motion.
What sort of questions?? =>
If you have a fire hose with some pressure, what floor will it reach?
If you have viscous flow through a tube, what is the velocity profile?
If you have flow over a wing, what is the lift? Drag?
To answer these questions we invoke Conservation Laws
What is conserved??
Mass: What goes in - what goes out = accumulation!
Momentum: Newton's 2nd Law of Motion:
\[ F = ma \]
Energy: First Law of Thermo!

We'll use these conservation laws to derive eqns that govern fluid motion, then apply to problems.

To do this, need a mathematical framework to describe motion.

Two approaches: Lagrangian & Eulerian

1) **Lagrangian**: Follow a fluid element as it moves thru flow:
\[ \mathbf{u} = \mathbf{u}(\mathbf{x}_0, t) = \mathbf{u}(\mathbf{x}_0; t) \]

Initial position $\mathbf{x}_0$; time $t$.

2) **Eulerian Approach**: \[ \mathbf{u} = \mathbf{u}(\mathbf{x}, t) \]
Track velocity field at an instant of time relative to defined coord system.

Ex: If you take a snapshot of a highway at time $t$, you could determine the velocity of all the cars, but you wouldn't know where they came from or where they wind up!

Both Eulerian & Lagrangian descr can provide a complete descr of flow, but for most fluid problems, Eulerian is more convenient—well focus on it!

Other useful concepts:
Streamline, Pathline, Streakline.

Also, \[ \mathbf{x} = \mathbf{x}(\mathbf{x}_0; t) = \mathbf{x}_0 + \int_0^t \mathbf{u}(\mathbf{x}_0; t') dt' \]

which tracks the position of the fluid element starting at $\mathbf{x}_0$ at $t=0$ for all time.

Lagrangian description isn't used much in fluids — a bit awkward! When would it be used?  $\Rightarrow$ celestial mechanics! Descr, positions of bodies (discrete) as $\mathbf{f}(t)$.

Also—study of suspensions (simulation) — track all the particles in a suspension!

$\Rightarrow$ Also important in pasteurization/related processes.

Streamline: curve everywhere tangent to velocity vector at a given instant $\Rightarrow$ a snapshot of the flow pattern!

$\Rightarrow$ this is what you get from Eulerian analysis.

Pathline: Actual path traversed by a given fluid element — Lagrangian description!

$\Rightarrow$ what you would get from time-lapse photograph of a marker in a flow field.

Streakline: locus of particles passing thru a given point

$\Rightarrow$ what is usually produced in flow visualization experiments; smoke is released continuously at a point, a pattern is photographed later.

For S.S. flow, all are identical!
Some unsteady flows may be made steady by shifting co-ords.

Example: Falling sphere in viscous fluid. It's moving w.r.t. inertial reference frame, so flow is unsteady. If we shift co-ord system so it travels with sphere, it's steady.

⇒ much more convenient mathematics as we eliminate time!

⇒ Note: we must use a constant velocity co-ord system! If we accelerate co-ord system, leads to non-inertial ref. frame ⇒ adds a term to the equations!

⇒ Also, flow past sphere may still be unsteady at higher Re due to vortex shedding, turbulence.

Another Concept: Control Volume

⇒ You used this in 255, etc.

⇒ Useful for deriving equations:

⇒ treat it as a “black box”, keeping track of what goes in & what goes out!

For example: What is the force on a pipe elbow? (50)

⇒ Just do a momentum balance!

⇒ Force = momentum out - momentum in!

(remember - momentum & force are vectors!)

Exerts force diagonal to elbow → why elbows need bracing!

What is flux thru face? (52)

\[ \text{Volumetric flux} = \dot{V} = \frac{\text{Vol}}{\text{Area} \times \text{Time}} \]

Mass flux \( \dot{m} = \frac{\text{Mass}}{\text{Area} \times \text{Time}} \)

Mass flux thru surface is proportional to component of \( \dot{m} \) (a vector) normal to the surface!

Since \( \dot{m} \) \( \neq 0 \), fluid ( & mass ) may come in ( or out ) thru each face!

\[ \text{So mass flow in thru these faces is:} \]
\[(g_y)_{x=10} - (g_y)_{x=0} \]

And if we combine this with the other faces:

\[
\begin{align*}
\text{Mass into cube} &= \left[(g_y)_{x=10} - (g_y)_{x=0}\right]_{x=0} + \left[(g_x)_{y=10} - (g_x)_{y=0}\right]_{y=0} + \left[(g_x)_{z=10} - (g_x)_{z=0}\right]_{z=0} \\
&= \frac{\partial}{\partial t} (ax \delta x \delta y \delta z) \\
&= \text{total mass!}
\end{align*}
\]

Dividing by \(ax \delta x \delta y \delta z\) & taking the limit as they go to zero yields:

\[
\frac{\partial\varphi}{\partial t} = -\left(\frac{\partial\varphi}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial\varphi}{\partial z}\frac{\partial g}{\partial z}\right)
\]

Remember the Lagrangian description:

\[
\frac{\partial\varphi}{\partial t} \text{ is the time rate of change of any property } \varphi \text{ experienced by a fluid element!}
\]

It has two components:
1) \(\frac{\partial\varphi}{\partial t}\) - local deriv. w.r.t. time
2) \(\mathbf{u} \cdot \nabla \varphi\) - change due to convection thru a field where \(\varphi\) varies with position

If a fluid is incompressible we have \(\varphi = \text{cst}\)

Thus \(\frac{\partial\varphi}{\partial t} = 0\)

and thus \(\mathbf{u} \cdot \nabla \varphi = 0\)

or, \(\frac{\partial\varphi}{\partial t} = -\nabla \cdot (\mathbf{g} \varphi)\)

In words: The time rate of change of the density is the negative of the divergence of the mass flux vector!

We can rearrange this:

\[
\frac{\partial\varphi}{\partial t} = -\nabla \cdot (\mathbf{g} \varphi)
\]

or \(\frac{\partial\varphi}{\partial t} + \mathbf{u} \cdot \nabla \varphi = -\nabla \cdot (\mathbf{g} \varphi)\)

This is known as the material derivative

\[
\frac{\partial\varphi}{\partial t} + \mathbf{u} \cdot \nabla \varphi = \nabla \cdot (\mathbf{g} \varphi)
\]

for any \(\varphi\)!

An alternate derivation may be made using vector calculus. Consider an arbitrary control volume \(D\):

What is the change in the total mass in \(D\)?

\[
\frac{\partial M}{\partial t} = \frac{\partial}{\partial t} \int_D \varphi \, dV = \frac{\partial}{\partial t} \int_D 2g \, dV
\]

\[
= \int_{\partial D} \varphi \mathbf{n} \cdot d\mathbf{A}
\]

\(\Rightarrow\) mass flux thru each patch of surface!
Thus:
\[ \sum_{D} \frac{\partial}{\partial x} \psi + \sum_{D} \psi \cdot n \, dA = 0 \]

Apply Divergence theorem:
\[ \frac{\partial}{\partial x} \psi + \nabla \cdot (\psi \mathbf{y}) = 0 \]

or
\[ \frac{\partial}{\partial x} + \nabla \cdot (\psi \mathbf{y}) = 0 \]

Which is the same equation!

In index notation:
\[ \frac{\partial \psi}{\partial x} + \psi \frac{\partial y}{\partial x} = 0 \]

To get the flow rate we use the CE:
\[ \frac{\partial}{\partial x} + \frac{\partial}{\partial x} (\psi \mathbf{y}) = 0 \]

We take the fluid to be incompressible, so the density is constant.
\[ \nabla \cdot \mathbf{y} = 0 \]

We draw a control volume:

Then:
\[ \sum_{v} \mathbf{y} \cdot dA = \sum_{v} \mathbf{y} \cdot n \, dA = 0 \]

\[ \sum_{v} \mathbf{y} \cdot dA = \sum_{v} \mathbf{y} \cdot n \, dA = 0 \]

Thus the average velocity:
\[ \langle u_e \rangle = \frac{A_p}{A_e} u_p \]
So the ratio of the average inlet velocity to the average outlet velocity is inverse of the ratios of the areas.  
Note: the CE tells you about the average velocity normal to the exit, it doesn’t tell you about the velocity distribution.

If there’s no flow, what is the pressure at the exit?

\[
P_e = \frac{F_A}{A_e} = \frac{M_3}{\pi r_p^2}
\]

\[P_e = P_o + P_{atm}
\]

What is the force required to raise the piston?

\[F = (P_e - P_{atm})A_e
\]

\[= \left(\frac{M_3}{\pi r_p^2} + \rho g h\right)\pi r_e^2
\]

\[= M g \frac{r_e^2}{r_o^2} + \pi r_e^2 \rho g h
\]

\[\text{small (usually)} \quad \rightarrow \text{ratio reduces required force!}
\]

This is how hydraulics work!

Examples: Car brakes, wing elevators, hydraulic jacks, etc.

Note: Energy expended to raise car is unchanged, but force is reduced!

Let’s extend the CE to multi-component systems.

Suppose we have \( m \) species, (e.g., salt sol’n: \( H_2O, NaCl: m=2 \))
we can do a balance on each species:

\[\text{Let velocity of species } i \text{ be given by } \vec{v}_i \text{ (or, not index notation here - subscript represents which species we’re talking about.)}
\]

\[\text{Note: v_i will, in general, be different from mass avg. velocity \( \vec{v} \) due to diffusion!}
\]

\[\text{Let density of species } i \text{ (mass/vol) be } \rho_i \quad \text{Note this is not the}
\]

\[\frac{\partial}{\partial t} \int_V \rho_i \, \text{d}V = -\int_V \rho_i \vec{v}_i \cdot \text{d}A + \int_D \text{R}_i \, \text{d}V
\]

\[\text{R}_i \Rightarrow \text{mass rate of production per unit volume of species } i \text{ due to reaction!}
\]

We can apply divergence theorem to this:
\[ \frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{u}_i) = 0 \]

or the microscopic \( \frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{u}_i) = R_i \)

The total density is just the sum of \( \rho_i \):

\[ \rho = \sum \rho_i \]

\& mass avg velocity:

\[ \mathbf{u} = \frac{1}{\rho} \sum \rho_i \mathbf{u}_i \]

Thus summing the equation over all species:

Suppose we have a well-mixed (stirred) tank:

We have a mass flow rate \( Q \):

\[ Q^{(i)} = \text{inlet mass flow} \]
\[ Q^{(e)} = \text{exit mass flow} \]

\[ M = \text{mass in tank} = \sum \rho \, \delta V \]

\[ \delta V = \text{control volume} \]

\[ \sum \rho_i \delta V = \text{mass in tank} \]

\[ \delta V = \text{density of salt} \]

\[ \delta V = \text{density of salt} \]

\[ \frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{u}_i) = \sum R_i \]

Note that \( \sum R_i = 0 \) since mass is conserved in reacting systems!

Next semester you will combine this equation with Fick's law to get the equation governing mass transfer!

OK, let's work another example: Conservation of mass in a CSTR (continuously Stirred Tank Reactor)

We wish to determine the fluid level & salt concentration as a function of time!

Thus:

\[ \frac{QM}{dt} = -\int_{\Omega} \nabla \cdot (\rho \mathbf{u}) \, dA \]

\[ = Q^{(i)} - Q^{(e)} \]

\[ \frac{\partial}{\partial t} \left( \sum \rho_i \right) = -\int_{\Omega} \nabla \cdot (\sum \rho_i \mathbf{u}) \, dA \]

\[ = Q^{(i)} \sum \frac{\partial \rho_i}{\partial t}^{(i)} - Q^{(e)} \sum \frac{\partial \rho_i}{\partial t}^{(e)} \]

\[ \omega_i^{(i)} \Rightarrow \text{mass fraction at inlet} \]
Now for a CSTR:
\[
\frac{S^{(e)}}{S^{(e)}} = \frac{S}{M} \quad (\text{tank is well mixed})
\]

Hence
\[
\frac{dS}{dt} = Q^{(i)} \omega_S^{(i)} - \left(\frac{S}{M}\right)Q^{(e)}
\]
\[
\frac{dM}{dt} = Q^{(i)} - Q^{(e)} = \Delta Q
\]

Solution: Solve for \(M\) first, then solve for \(S\):
\[
M = M_0 + \Delta Q t \quad (\text{linear change in time})
\]
\[
\frac{dS}{dt} = -\frac{Q^{(e)}}{M_0 + \Delta Q t} S + Q^{(i)} \omega_S^{(i)}
\]
\[
p(x) = \frac{Q^{(i)} \omega_S^{(i)}}{M_0 + \Delta Q t}
\]
\[
\int p(x)dx = \frac{Q^{(e)}}{\Delta Q} \ln \left(\frac{M_0}{\Delta Q} + t\right)
\]

And thus:
\[
S = \left[ \int Q^{(i)} \omega_S^{(i)} e^{-\left(\frac{Q^{(e)}}{\Delta Q} \ln \left(\frac{M_0}{\Delta Q} + t\right)\right)} dt + K \right]
\]
\[
time e^{-\left(\frac{Q^{(e)}}{\Delta Q} \ln \left(\frac{M_0}{\Delta Q} + t\right)\right)} \quad \text{and}
\]
\[
S = \left(e^{Q^{(e)} \ln \left(\frac{M_0}{\Delta Q} + t\right)} \right) - Q^{(i)} \omega_S^{(i)} \left(e^{-\left(\frac{Q^{(e)}}{\Delta Q} \ln \left(\frac{M_0}{\Delta Q} + t\right)\right)} \right)
\]

Thus:
\[
S = \left(\frac{M_0}{\Delta Q} + t\right) Q^{(e)} \left[ Q^{(i)} \omega_S^{(i)} + \left(e^{Q^{(e)} \ln \left(\frac{M_0}{\Delta Q} + t\right)} - 1\right)\right]\]
\[
\left(\frac{M_0}{\Delta Q} + t\right) Q^{(e)} \left[ Q^{(i)} \omega_S^{(i)} \frac{\left(e^{Q^{(e)} \ln \left(\frac{M_0}{\Delta Q} + t\right)} - 1\right)}{\left(e^{Q^{(e)} \ln \left(\frac{M_0}{\Delta Q} + t\right)} + 1\right)} + K\right]
\]

or
\[
\frac{dS}{dt} + \frac{S}{M_0 + \Delta Q t} \frac{dS}{dt} + \frac{S}{M_0 + \Delta Q t} \frac{dS}{dt} = Q^{(i)} \omega_S^{(i)}
\]

By I.C. \(S\big|_{t=0} = S_0\)

This is a first order linear ODE

We have the general solution
\[
\frac{dy}{dx} + p(x) y = f(x)
\]

Then:
\[
\int p(x)dx \left[ \int \frac{f(x)}{y^{(*)}} dx + K \right]
\]

where \(K\) is determined from I.C.

Let's apply this:
\[
x = t, \quad f(x) = Q^{(i)} \omega_S^{(i)} = C t
\]

\[
S = C t \left(\frac{M_0}{\Delta Q} + t\right) \left(e^{Q^{(e)} \ln \left(\frac{M_0}{\Delta Q} + t\right)} - 1\right) + K \left(\frac{M_0}{\Delta Q} + t\right)
\]

We determine \(K\) from the I.C.
\[
S\big|_{t=0} = S_0
\]

Thus:
\[
S_0 = Q^{(i)} \omega_S^{(i)} \left(\frac{M_0}{\Delta Q} + t\right) + K \left(\frac{M_0}{\Delta Q} + t\right)
\]

So:
\[
K = S_0 \left(\frac{M_0}{\Delta Q} - Q^{(i)} \omega_S^{(i)} \left(\frac{M_0}{\Delta Q} + t\right)\right)
\]

Which yields:
\[
S = S_0 \left(\frac{M_0}{\Delta Q} + t\right) + Q^{(i)} \omega_S^{(i)} \left(\frac{M_0}{\Delta Q} + t\right)
\]

\[
\left[ Q^{(i)} \omega_S^{(i)} \frac{\left(e^{Q^{(e)} \ln \left(\frac{M_0}{\Delta Q} + t\right)} - 1\right)}{\left(e^{Q^{(e)} \ln \left(\frac{M_0}{\Delta Q} + t\right)} + 1\right)} + K\right]
\]

\[
\left(\frac{M_0}{\Delta Q} + t\right) - \left(\frac{M_0}{\Delta Q} + t\right) \left(e^{Q^{(e)} \ln \left(\frac{M_0}{\Delta Q} + t\right)} + 1\right)
\]
We can simplify a bit further if we recall:

\[ M = M_0 + \Delta Q t \]

Thus:

\[ S = S_0 \left( \frac{M_0}{M} \right)^{\frac{\rho_0}{\rho}} + \frac{\rho_0}{\rho} \left( \frac{M_0}{M} \right)^{\frac{\rho_0}{\rho}} \]

It is interesting to note that in the limit \( \Delta Q \to 0 \) (e.g., \( Q^{(0)} \to Q^{(i)} \)) the power law form given here collapses to a pure exponential:

\[ S = S_0 (M_0/M)^{\frac{\rho_0}{\rho}} + \frac{\rho_0}{\rho} \left( \frac{M_0}{M} \right)^{\frac{\rho_0}{\rho}} \]

The quantity \( M/Q \) is known as the **Residence Time** of the vessel.

What do these terms look like?

\[ \dot{g} = \text{momentum per unit volume} \]

Thus:

\[ \dot{g} = \frac{\text{rate of momentum out}}{\text{by convection}} \]

\[ \dot{g} = \frac{\text{momentum}}{-\text{volume}} \times \text{normal to surface} \]

What is the total momentum in \( D \)?

\[ g \hat{g} = \text{momentum per volume} \]

Thus accumulation is:

\[ \int_{\partial D} \rho \hat{g} (3g) \, dA \]
Combining these terms:

\[ \sum \frac{\partial (\rho u)}{\partial t} \, dV + \sum (\rho u) \cdot n \, dA = \sum F \] (sum of forces on control volume)

Ok, what are the forces? We looked at these before:
- Body forces (e.g., gravity):
  \[ F_1 = \sum \rho g \, dV \]
- Surface forces:
  - These include normal forces (e.g., pressure) and shear forces.
  - The latter results from "dragging" along (tangential to) a surface!

\[ \sum (\rho u) \cdot n \, dA = \sum F \label{eq:forces} \]

How can we use this? \( \Rightarrow \) we can calculate the force on an elbow!

Suppose we know inlet & outlet pressures as well as the flow rate. We want to know the force exerted by the fluid on the bend (section of pipe) which is \( \Rightarrow \) force exerted by bend on fluid.

Let \( F \) be all surface forces at a point.

Thus:

\[ \sum F = \sum \rho g dV + \sum F dA \]

\[ F = \text{force} \cdot \text{area} \]

Recall from our earlier examination of hydrostatics that:

\[ F = \text{stress} \cdot \text{area} \]

where \( \text{stress} \) is the stress tensor we'll use this in a bit. For now we have:

\[ \sum \rho g dV + \sum F dA = \sum (\rho u) \cdot n \, dA \]

We have the momentum balance:

\[ \sum \rho g dV + \sum F dA = \sum (\rho u) \cdot n \, dA \]

We assume we are at steady state, thus \( \frac{\partial}{\partial t} \equiv 0 \)

If the fluid is incompressible:

\[ \sum \rho g dV = \rho g V_0 = \text{weight of fluid} \]

Now for the surface integrals:

We divide up \( dA \) into \( A_i, A_e, A_p \) (pipe area)

\[ A_i - A_e \rightarrow A_p \]
Let's look at the convection term:

\[
\int_{A_D} (u \cdot \hat{n}) \, dA = \int_{A_I} (u \cdot \hat{n}) \, dA + \int_{A_L} (u \cdot \hat{n}) \, dA + \int_{A_E} (u \cdot \hat{n}) \, dA
\]

Over the pipe itself \((A_I)\) \(u \cdot \hat{n} = 0\) (no flow through the pipe), thus we just get integrals over inlet & exit.

\[
\Rightarrow \text{Unlike mass conservation, we can't evaluate integrals exactly without knowing the velocity profile \(u(y)\) across the pipe in addition to the total flow rate \(Q\)}
\]

This is because the integral is non-linear in \(u\)!

This is because non-uniformities in \(u\) increase the momentum flux over a uniform velocity!

\[
\Rightarrow \text{The average of the square is always greater than or equal to the square of the average!}
\]

Let \(u = \frac{1}{A} \int_A u \, dA\)

Let \(du = u - \langle u \rangle\)

So:

\[
\int_A u^2 \, dA = \int_A (du + \langle u \rangle)^2 \, dA + \int_A \langle u \rangle^2 \, dA
\]

\[
= \int_A u^2 \, dA + \int_A (\langle u \rangle)^2 \, dA + \int_A \langle u \rangle \, dA
\]

Thus:

\[
\int_A (u \cdot \hat{n}) \, dA = \int_A (\langle u \rangle \cdot \hat{n}) \, dA + \int_A (u - \langle u \rangle) \cdot \hat{n} \, dA
\]

So we get:

\[
-\int_A \langle u \rangle \cdot \hat{n} \, dA = -\int_A \langle u \rangle \cdot \hat{n} \, dA
\]

To estimate the force we shall assume we have uniform flow

Let's take \(u \mid_{A_I} = \frac{\sqrt{2}}{A_I} \hat{x}\)

Now at the inlet \(\hat{n} \mid_{A_I} = -\hat{x}\)

Thus:

\[
\int_{A_I} (u \cdot \hat{n}) \, dA = \int_{A_I} \langle u \rangle \cdot \hat{n} \, dA = \int_{A_I} (\langle u \rangle \cdot \hat{n}) \, dA
\]

\[
\hat{e}_x \cdot (\hat{e}_x) = -1
\]

So we get:

\[
-\int_A \langle u \rangle \cdot \hat{n} \, dA = -\int_A \langle u \rangle \cdot \hat{n} \, dA
\]

which is negative because momentum is going into CV!  

Note that this underestimates the momentum flux (in general).

Since \(\int_A (u \cdot \hat{n}) \, dA \geq 0\)

we have:

\[
\int_A u^2 \, dA \geq \int_A \langle u \rangle^2 \, dA
\]

so we underestimate the momentum flux. For high Re (turbulence) the profile is nearly flat (uniform), so it's not a big error!

Over the exit we have the same integral:

\[
\hat{u} \bigg\rvert_{A_e} = \frac{\sqrt{2}}{A_e} \hat{e}_x, \quad \hat{e}_o = \cos \theta \hat{x} + \sin \theta \hat{y}
\]

The unit normal: \(\hat{n} \mid_{A_e} = \frac{\hat{e}_x}{A_e}\)

So:

\[
\int_{A_e} (u \cdot \hat{n}) \, dA = \int_{A_e} \frac{u^2}{A_e} \hat{e}_x \cdot \hat{e}_x
\]
Putting these together:
\[ \sum_{\text{area}} q_x q_y \text{d}A \approx \frac{Q^2}{A_i} \left( \frac{\hat{e}_x}{A_i} + \frac{\hat{e}_y}{A_e} \right) \]

Note that since \( \hat{e}_x \neq \hat{e}_y \) the force will be non-zero even if \( A_i = A_e \).

A force is required to deflect a stream!

Ok, now we look at the surface forces:

\[ \sum_{\text{area}} q_x \text{d}A \equiv \sum_{A_i} \frac{f_x}{A_i} + \sum_{A_e} \frac{f_y}{A_e} \]

The last one is what we’re after! Let’s do the first term:

\[ \sum_{A_i} f_x \text{d}A \equiv \text{force exerted by fluid outside CV on EV integrated over } A_i \]

Putting it all together:

\[ \frac{\partial}{\partial \nu} \left( \frac{f_x}{A_i} \right) \text{d}A \bigg|_0 = \sum_{\text{area}} q_x q_y \text{d}A \]

or:

\[ q_x q_y \text{d}A \bigg|_0 = \frac{Q^2}{A_i} \left( \frac{\hat{e}_x}{A_i} + \frac{\hat{e}_y}{A_e} \right) = -gqV_0 \hat{e}_y + P_i A_i \hat{e}_x - P_e A_e \hat{e}_y + F_p \]

or, rearranging:

\[ F_p = gq^2 \left( \frac{\hat{e}_x}{A_i} + \frac{\hat{e}_y}{A_e} \right) + gqV_0 \hat{e}_y - P_i A_i \hat{e}_x + P_e A_e \hat{e}_y \]

This is a vector equation! We can look at the \( x \) component:

\[ (F_p)_x = F_p \cdot \hat{e}_x = gq^2 \left( \frac{-1}{A_i} + \frac{\cos\theta}{A_e} \right) - P_i A_i + P_e A_e \cos\theta \]

or the \( y \) component:

\[ (F_p)_y = F_p \cdot \hat{e}_y = gq^2 \left( \frac{-\sin\theta}{A_e} \right) + gqV_0 - P_e A_e \sin\theta \]

These forces could be used to determine the required bracing, for example!
Let's work through another example: Water jet pushing a car. Suppose we have a car with a plate sticking up as below:

\[ \begin{array}{c}
\text{Jet} \\
\text{(U, V)}
\end{array} \]

A jet of water of diameter D & velocity \( U_j \) impinges on the plate. What is the force on the plate as a function of \( U \)? What is the velocity of the car as a function of time?

To solve, look at problem in a reference frame moving with the plate!

Thus:

\[ F_x = A \left( \frac{g}{j} \left( U_j - U \right) \right) \]  \( \frac{dx}{dt} \)

So the force on the fluid is just:

\[ F_x = -Ag \left( U_j - U \right)^2 \]

The force on the car is the negative of this!

Now since \( F = M \frac{dU}{dt} \), we have:

\[ \frac{dU}{dt} = \frac{Ag}{M} \left( U_j - U \right)^2 \]

We can solve this:

\[ \frac{1}{U - U_j^2} \frac{dU}{dt} = \frac{Ag}{M} \]

Water velocity in this frame is now \( (U_j - U) \), not \( U_j \)!

We draw the CV as depicted. we have:

\[ \sum F = \int \left( \mu \frac{dV}{dt} \right) A \frac{dA}{dV} \]

We are interested in the \( x \)-component of this force. Since the fluid leaves \( BD \) with a velocity only in the \( y \)-direction, we just worry about the inlet:

\[ \frac{1}{U - U_j} \frac{dU}{dt} = -\frac{Ag}{M} \]

Let \( U_j = 0 \), \( t \to 0 \):

\[ C = -\frac{1}{U_j} \]

So:

\[ \frac{1}{U - U_j} = -\frac{Ag}{M} t + C \]

\[ U_j = 1 - \frac{1}{\frac{Ag}{M} t + 1} \]

Thus:

\[ \frac{1}{U - U_j^2} \frac{dU}{dt} = \frac{Ag}{M} \]

So \( U \) asymptotically approaches \( U_j \) as we would expect.
We cannot get a much higher force & acceleration if we modify the plate so it sends water back out in the reverse direction.

In the moving reference frame we still have:
\[ \sum F = \int (\vec{F}) \cdot d\vec{A} \]
but now \( u_x \) is reversed for the fluid leaving 2D rather than just zero. This doubles the momentum transfer!

Let's analyze the Pelton wheel:

\[ u_x = \frac{Q}{2} R \]

we wish to determine the torque on the wheel, and the rate of work (power) transferred to it.

For the torque:
\[ M = \vec{F} \cdot \vec{R} \]
The force is just the change in momentum of the stream! To get this, we need the exit velocity \( u_x \). We have the two cases for different vanes: flat plate & reflection.

\[ \begin{align*}
F_x &= -2 A \frac{g}{2} (u_x - u)^2 \\
&= \text{force on fluid} \\
\frac{du}{dt} &= 2 \frac{A g}{M} (u - u_x)^2 \\
or \frac{u}{u_x} &= \frac{2 A \delta u t}{1 + 2 A \delta u t} \\
The asymptotic velocity is still \( u_x \), it just gets there twice as fast! \\
This effect is why Pelton wheel (a type of turbine) the buckets are curved & more efficient momentum & energy transfer.
\end{align*} \]

\[ F = \frac{Q}{2} (u_x - u) \frac{d^2 u}{d t^2} \]
\( u_x \) & \( u \) mean vane \( x \) & \( y \) \\
Force on vane (neg. of force on fluid) \\
Only the \( x \)-component of the force contributes to the torque! (prop. to \( R \))

For the flat plate we have:
\[ \begin{align*}
\vec{F} &= \frac{Q}{2} (u_x - u) \frac{d^2 u}{d t^2} R \\
\text{Thus for this case} \\
F_x &= Q \left[ (u_x - u) \frac{d^2 u}{d t^2} \right] \\
The torque is \( F_x R \). \\
What about the power? \\
P &= M \cdot \omega = F_x R \omega \\
= Q R \omega \left[ (u_x - u) \frac{d^2 u}{d t^2} \right] \]
Note that the torque is zero when \( \omega = 0 \); the power is zero.

What is the value of \( \omega \) for which the power is max? \( \frac{dP}{d\omega} = 0 \), \( 2 \omega U_2 = \omega R \left[ 8U_2 - 2gU_2R \right] \)

\( U_2 = 2gU_2R \)

or \( \omega \frac{R}{2} = \frac{U_2}{2} \)
	now the vane moves with half the velocity of the jet. The max power is:

\[
P_m = \frac{1}{2} Q \left( \frac{1}{2} \frac{U_2}{2} \right) \]

which is twice the force (and torque and power) at the flat vane.

At the optimum (same) rotation rate, we have:

\[
P_m = Q \left( \frac{1}{2} U_2^2 \right) \]

or all the kinetic energy of the jet is extracted. A real water wheel would lie between these values.

Microscopic Momentum Balances

So far we've done our calculations by assuming velocity profiles were flat (uniform). This, in general, is not correct! To get it right, we need to calculate the velocity profile. We need to develop the equation which governs the velocity everywhere in the fluid.

To do this, we need to reexamine the stress tensor \( \tau \).

Look at the flow between parallel plates:

\[
\begin{align*}
U_x \quad F \\
\uparrow \quad \downarrow \\
Y \quad x
\end{align*}
\]

Fluid resists deformation so a force \( F \) is required to keep the plate in motion!

The magnitude of the force is proportional to the Area, thus we look at \( \tau A \Rightarrow \text{shear stress} \)

at the wall.

Shear stress is transmitted through the fluid to the lower plate!

\( \text{Shear stress} = \text{momentum flux} \)

For this geometry, each layer of fluid exerts the same force on the layer below it! The shear stress is constant, otherwise momentum would accumulate in the interior.
Recall the definition of $\sigma_{ij}$:

$$\sigma_{ij} = F/A$$

...exerted by fluid of greater i on fluid of lesser j in direction.

In this case, we have:

$$\sigma_{yx} = F/A$$

Which, for this geometry, is constant.

What are the properties of $\sigma_{ij}$?

$\sigma_{ij} = \sigma_{ji}$

$\sigma_{ij} = \sigma^{-1}_{ji}$

This is really counter intuitive.

In this flow:

$$\begin{align*}
\mathbf{F}_x &= \mathbf{F}_y \\
\mathbf{F}_z &= \mathbf{F}_x \\
\mathbf{F}_y &= \mathbf{F}_z \\
\mathbf{F}_z &= \mathbf{F}_y
\end{align*}$$

Now we have $\Sigma F = 0$ because element isn’t accelerating.

What about the torque?

$$M = \sum \mathbf{r} \times \mathbf{F} = -\frac{\partial}{\partial z} \mathbf{T}_{xy} (\partial z \mathbf{e}_z)$$

$$-\frac{\partial}{\partial z} \mathbf{T}_{xy} (\partial z \mathbf{e}_z) = \frac{\partial}{\partial z} \mathbf{T}_{yx} (\partial z \mathbf{e}_z)$$

$$\frac{\partial}{\partial z} \mathbf{T}_{xy} (\partial z \mathbf{e}_z)$$

$$\frac{\partial}{\partial z} \mathbf{T}_{yx} (\partial z \mathbf{e}_z)$$

We have, just like $F = ma$, a relation for the angular acceleration of any object:

$$\frac{d^2 \omega}{dt^2} = \frac{\mathbf{M}}{I}$$

$M$ = moment of inertia

$$I = \int_0^R s^2 d\mathbf{r} = \frac{axa^2}{12} \mathbf{s} (ax^2 ay^2)$$

Thus:

$$\frac{d^2 \omega}{dt^2} = \frac{\mathbf{M}}{I} = \frac{axa^2}{12} \mathbf{s} \left( \mathbf{T}_{xy} - \mathbf{T}_{yx} \right)$$

As $ax, ay \to 0$ any angular acceleration must be finite,

thus we conclude $\mathbf{T}_{yx} = \mathbf{T}_{xy}$!

There is an exception to this: For very weird systems you can get a body torque when torque applied uniformly through a fluid. This would make the stress tensor asymmetric! How can you do this?

If you have an E.R. (electro rheological) fluid in a rotating electric or magnetic field you get this effect. Don’t worry about it! For all
normal systems, the stress tensor is symmetric!!

Another useful property:
For any surface \( S \) with normal \( \mathbf{n} \), the stress (force/area) exerted by surroundings on fluid is just:
\[
\mathbf{f} = \mathbf{T} \cdot \mathbf{n}
\]

We can use this in our momentum balance equations:
Recall:
\[
\begin{align*}
\{ \text{not momentum out} \} + \{ \text{Accumulation} \} = \\
\{ \text{body forces} \} + \{ \text{surface forces} \}
\end{align*}
\]

We can simplify this by differentiating by parts:
\[
\nabla \cdot (\mathbf{g} \mathbf{u} \mathbf{n}) \equiv \mathbf{g} \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla (\mathbf{g} \cdot \mathbf{n})
\]
\[
\frac{\partial (\mathbf{g} \cdot \mathbf{n})}{\partial t} = \mathbf{u} \cdot \frac{\partial \mathbf{g}}{\partial t} + \mathbf{g} \cdot \frac{\partial \mathbf{u}}{\partial t}
\]
Substituting in:
\[
\begin{align*}
\frac{\partial \mathbf{g}}{\partial t} + \mathbf{u} \cdot \frac{\partial \mathbf{g}}{\partial t} &+ \mathbf{u} \left[ \frac{\partial (\mathbf{g} \cdot \mathbf{n})}{\partial t} + \nabla \cdot (\mathbf{g} \mathbf{n}) \right] \\
&= \mathbf{g} \cdot \frac{\partial \mathbf{u}}{\partial t} + \mathbf{g} \cdot \mathbf{n}
\end{align*}
\]

Now from conservation of mass:
\[
\frac{\partial \mathbf{g}}{\partial t} = -\nabla \cdot (\mathbf{g} \mathbf{u})
\]
thus the term in brackets is zero
So:
\[
\mathbf{g} \left[ \frac{\partial (\mathbf{g} \cdot \mathbf{n})}{\partial t} + \nabla \cdot (\mathbf{g} \mathbf{n}) \right] = \mathbf{g} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{g} \cdot \mathbf{n}
\]
Or:
\[
\frac{\partial \mathbf{g}}{\partial t} = \mathbf{g} \cdot \frac{\partial \mathbf{u}}{\partial t} + \mathbf{g} \cdot \mathbf{n}
\]
We can also write this in index notation:
\[
\mathbf{g} \left( \frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x_j} \right) = \frac{\partial g_j}{\partial x_i} + \mathbf{g} \cdot \mathbf{n}
\]
Note that each term has only one unrepeated index, and that they are all the same!

To proceed, we look at the total stress \( \mathbf{\tau} \) We define:
\[
\tau_{ij} = -p \delta_{ij} + \tau_{ij}
\]
where \( p = -\frac{1}{3} \left( \sigma_{ii} + \sigma_{jj} + \sigma_{kk} \right) \) in the pressure - the average of the normal stresses in the three orthogonal directions (with the negative of this anyway)
Other ways of saying this:
\[ \tau_{ij} = -\frac{1}{2} \sigma_{ii} \delta_{ij} \]
where \( \sigma_{ii} = \text{trace} (\sigma) \)
\( \tau_{ij} \) is known as the deviatoric stress and arises due to fluid motion. It is identically zero for isotropic fluids at rest (e.g., hydrostatics).

What are the properties of \( \tau_{ij} \)?
- Since \( \delta_{ij} \) is symmetric, so is \( \tau_{ij} \)
- By definition, \( \tau_{ij} \) is traceless
  \[ \tau_{ij} \delta_{ij} = 0 \]
  \[ \tau_{ij} \delta_{ij} = \delta_{ij} \tau_{ij} + \delta_{ij} \delta_{ij} \tau_{ij} \]
  \[ \tau_{ii} = \frac{\sigma_{ii}}{3} = 0 \]

We can generalize this a bit:
Remember that \( \delta_{ij} \) is symmetric.
Thus \( \tau_{yx} = \tau_{xy} = \mu \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \).

Actually, we can generalize this still further. If \( \tau_{ij} \) is proportional to the rate of strain tensor \( \frac{\partial u_i}{\partial x_j} \), we have the general relation:
\[ \tau_{ij} = A_{ik} \delta_{kj} \frac{\partial u_i}{\partial x_j} \]
where \( A_{ik} \) is a fourth order tensor. We have three restrictions on \( A_{ik} \): First, if the fluid is isotropic, then \( A_{ik} \) must also be isotropic (it's a material property).

\[ \Rightarrow \tau_{ij} \text{ arises from the deformation of a fluid!} \]

As an example, consider flow between two parallel plates:
\[ \begin{array}{c}
\gamma \\
\hline
\end{array} \]

In this geometry, \( \tau_{yx} = \frac{F}{A} \)
Experimentally, we find:
\[ F = \mu \frac{\partial u}{\partial y} \]
where \( \mu \) is the fluid viscosity!

Now we also have:
\[ \frac{\partial u}{\partial y} = \frac{Q}{h} \]
(linear profile)

Thus we get Newton's Law of Viscosity:
\[ \tau_{yx} = \mu \frac{\partial u}{\partial y} \]

Thus:
\[ A_{ijk} = \lambda \delta_{ij} \delta_{kl} + \lambda_2 \delta_{ik} \delta_{jl} + \lambda_3 \delta_{ij} \delta_{lk} \]

Secondly, we know that \( \tau_{ij} \) is symmetric, e.g. that \( \tau_{ij} = \tau_{ji} \)
This requires \( A_{ijk} = A_{ikj} \) or that \( \lambda_2 = \lambda_3 \)
Finally, we know that \( \tau_{ij} \) is traceless, e.g. that \( \tau_{ij} \delta_{ij} = 0 \)
This requires \( \delta_{ij} A_{ prop} = 0 \)
Plugging this in, we get \( \lambda_1 = -\frac{2}{3} \lambda_2 \)
Thus:
\[ A_{ijk} = \lambda_2 \left[ \delta_{ik} \delta_{jl} + \delta_{ij} \delta_{lk} - \frac{2}{3} \delta_{il} \delta_{jk} \right] \]
or, as it's usually written:
\[ \tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_j}{\partial x_j} \delta_{ij} \nabla \cdot \mathbf{u} \right) \]

So we see that this complex expression for the shear stress arises naturally from the assumptions of linearity, isotropy, and the definition of the pressure (\( \tau_{ii} = 0 \)).

For more complex fluids, the stress-strain relation is a bit messier! The study of such relations is the field of rheology.

For an incompressible fluid
\[ \nabla \cdot \mathbf{u} = 0, \]
thus:
\[ \tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

We can plug this in:

\[ g \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = \frac{\partial \tau_{ij}}{\partial x_j} + \gamma g_i \]

Thus:
\[ g \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \gamma g_i \]

but:
\[ \frac{\partial \tau_{ij}}{\partial x_j} = \mu \left( \frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_i}{\partial x_i \partial x_j} \right) = \mu \left( \frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right) \]

So:
\[ g \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \gamma g_i \]

which are known as the Navier-Stokes equations. They are valid for

incompressible, Newtonian fluids with constant viscosity (or at least not a function of position!).

If any of these assumptions are not valid, the equations need to be modified! Fortunately, they work for most chemical problems!

Let's look at the equation term by term:
\[ \frac{\partial \mathbf{u}}{\partial t} \rightarrow \text{time dependent accumulation of momentum} \]
\[ \nabla \cdot \mathbf{u} \rightarrow \text{Convection of momentum, associated with fluid inertia} \]
\[ -\nabla P \rightarrow \text{Gradients in the pressure act as a source or sink of momentum} \]
\[ \mu \nabla^2 \mathbf{u} \rightarrow \text{Viscous diffusion of momentum} \]

\[ \gamma \rightarrow \text{Gravitational (body force) source of momentum} \]

Try to build up a physical picture of each of the physical mechanisms behind these terms! Such an understanding will help you determine which terms are important in any physical problem!

One, now let's apply these equations to the simplest flow problem:

**Plane Couette Flow**

\[ \mathbf{u} \]

\[ \begin{cases} \mathbf{u} \rightarrow & \mathbf{0} \quad \text{at } x = 0 \\mathbf{u} \rightarrow & (\mathbf{0}, \mathbf{0}) \quad \text{at } x = h \end{cases} \]

we assume an incompressible, Newtonian fluid with constant viscosity, thus
we have the equations:

\[ \begin{align*}
    \frac{\partial \rho}{\partial t} & = 0 \\
    \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} & = -\nabla P + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x
\end{align*} \]

We also need boundary conditions

\[ \begin{align*}
    u & = 0 \quad \text{(all 3 components)} \\
    y & = 0 \\
    u & = U_0 \hat{x} \quad \text{(y & z components are zero)} \\
    y & = h
\end{align*} \]

Now we start throwing out terms.

We anticipate that the flow is only in the x-direction, thus

\[ u_y = u_z = 0 \]

From continuity,

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0 \]

Thus \( \frac{\partial u}{\partial x} = 0 \)

\[ \Rightarrow \text{There is no change in the velocity in the flow direction for unidirectional flow.} \]

\[ \Rightarrow \text{The converse: If the velocity changes in the flow direction, then it cannot be unidirectional!} \]

\[ \text{Example, if } \frac{\partial u}{\partial x} \neq 0 \text{ then } u_y \text{ or } u_z \text{ must be non-zero somewhere} \]

We assume that the flow is 2-D (no change in z-direction), thus \( \frac{\partial u}{\partial z} = 0 \)

We assume that there are no applied pressure gradients, thus \( \frac{\partial P}{\partial x} = 0 \)

We take \( g_y = -g \hat{y} \), not in x-direction.

We assume flow is at steady-state, so \( \frac{\partial u}{\partial t} = 0 \)

Ok, what's left?

C.E.:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]

So \( \frac{\partial u}{\partial x} = 0 \)

\[ \Rightarrow \text{Momentum: } \]

\[ \begin{align*}
    \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} & = -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_y \\
    & = -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \rho g_y \\
    & = -\rho g_y \quad \text{(by assumption)} \\
    \Rightarrow & \text{So } \frac{\partial P}{\partial y} = \rho g_y \quad \text{(no pressure gradient in z-direction)} \\
\end{align*} \]

Hence \( P = f(x) - \rho g y \)

\[ \text{ Actually, will be a cost since no gradient is applied in x-direction} \]

Just by hydrostatic pressure variation!

Now for x-momentum (this is the important one, because the flow is in the x-direction!)}
No pressure gradient (applied) in x-direction, so
\[-\frac{\partial p}{\partial x} = 0\]

What’s left?
\[-\frac{\partial^2 u_x}{\partial y^2} = 0, \quad u_x |_{y=0} = 0\]
\[u_x |_{y=h} = U_0\]

This is easily solved - just integrate twice!
\[u_x = Ay + B\]
\[u_x |_{y=0} = 0 \quad \Rightarrow \quad B = 0\]
\[u_x |_{y=h} = U_0 \quad \Rightarrow \quad A = \frac{U_0}{h}\]

And
\[u_x = \frac{U_0 y}{h}\]

What’s this relationship? 

\[\Rightarrow \text{if } A R << 1 \text{ we can ignore curvature effects:}\]
\[U_0 \approx \frac{U_0}{h} = \frac{U_0}{A R}\]

Thus
\[u_x \approx \frac{U_0 y}{A R}\]

The stress on the inner cylinder is:
\[\tau_{yx} = \mu \frac{\partial u_x}{\partial y} = \mu \frac{U_0}{A R}\]

The torque is:
\[M = (\tau_{yx})(R) (2\pi RH)\]
where \(H\) is the height in the z-direction. Thus:
\[M = \mu \frac{A R^2 R^3 H}{2 R} \text{ for } A R << 1\]

which can be used to estimate \(\mu\)
Another example: Flow down an inclined plane.

If the fluid is viscous, it will rapidly reach some constant thickness $h$, and some steady velocity profile. What is the relationship between $Q, w, u, v, h, s, \mu, k$, and $\Theta$?? Just apply the Navier-Stokes equations!

First, we choose a coordinate system aligned with the geometry.

Important: crossing out those you expect to be zero. If you can satisfy all B.C.'s with the simplified equation, you get it right! This is strictly true only for linear problems, as non-linear equations often have multiple solutions. Even there, it's a good way to start.

Know each term physically.

Ok, we expect unidirectional flow. Thus:

$$0 = \frac{\partial^2 p}{\partial y^2} + s g_y$$

$$\frac{\partial}{\partial x} \left[ \mu \frac{\partial u}{\partial x} + s g_x \right] = 0$$

Let $x$ be the direction along the plate, and $y$ be normal to the plate $w | y=0$ at the plate:

Thus $\mathbf{g} = -g \cos \theta \mathbf{e}_y + g \sin \theta \mathbf{e}_x$

Again, we have unidirectional flow in the $x$-direction. We expect there will be no flow in the $y$-direction—just a hydrostatic pressure variation.

Note: to solve these sorts of problems, look at it physically & keep those terms which appear to recall that for unidirectional incompressible flow:

$$\frac{\partial u}{\partial x} = 0$$

There is no variation in the $z$-direction (2-D flow), thus

$$\frac{\partial^2 u}{\partial y^2} = 0$$

Now to solve. First we get the pressure distribution.

$$\mathbf{g} = -g \cos \theta$$

$$\therefore \mathbf{p} = f(x) - s g y \cos \theta$$

but $\mathbf{p}_{|y=0} = P_o$ (atmospheric)

Thus $\mathbf{p} = P_o + s g (s - y) \cos \theta$
This is an example of the effect of nonlinearities. There are multiple solutions to the set of equations, where \( \psi \neq 0 \), and if we were to study up to \( \psi = \psi_{\text{max}} \) (from the working point at \( \psi = 0 \), we see that \( \psi = \psi_{\text{max}} \) at \( \gamma = \frac{\pi}{2} \)) models with \( \psi = \psi_{\text{max}} \) (from the working point at \( \psi = 0 \), we see that \( \psi = \psi_{\text{max}} \) at \( \gamma = \frac{\pi}{2} \)).

From this we see that \( \psi > 0 \), and from this we see that \( \psi > 0 \).
Another example: Flow through a pipe!

Suppose we have an axial pressure gradient (e.g., \( \frac{\partial p}{\partial z} = 0 \)).

What is the flow profile?

For a given \( \rho \), \( \mu \), \( R \), what is the flow rate? Again we choose a coordinate system aligned with the boundary: cylindrical coordinates!

Let’s solve this: We begin with the C.B.:

\[
0 = \frac{1}{\rho} \frac{\partial}{\partial r} (\rho \mu \frac{\partial u}{\partial r}) + \frac{\mu}{\rho} g \frac{\partial u}{\partial z} = 0
\]

So:

\[
\mu \frac{1}{\rho} \frac{\partial}{\partial r} (\rho \mu \frac{\partial u}{\partial r}) = \frac{\partial p}{\partial z} - g \rho g = 0
\]

Note that there are two possible sources for momentum: pressure gradients or gravity. Both act in exactly the same way! Both (if constant) are uniform sources (or sinks) of momentum in the fluid. Here we take \( g = 0 \) and look at the pressure gradient.

Let \( \frac{\partial p}{\partial z} = \frac{\Delta p}{L} \) (pressure drop/length) (note: this is positive)

So: \( \frac{1}{\rho} \frac{\partial}{\partial r} (\rho \mu \frac{\partial u}{\partial r}) = \frac{\mu}{\rho} \frac{\Delta p}{L} = \text{const} \)

We integrate once:

For uni-directional flow in the \( z \)-direction, \( u_r = u_\theta = 0 \).

Thus \( \frac{\partial u_z}{\partial z} = 0 \)

The assumption of unidirectional flow will limit the applicability of our solution! We’ll see how this works later!

Ok, now we solve for the velocity distribution. We focus on the \( z \)-momentum epn in cylindrical form:

\[
\frac{\partial}{\partial r} \left[ \frac{1}{2} \rho \left( \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} \right) \right] + \frac{\mu}{\rho} \frac{\partial u_r}{\partial \theta} = \frac{1}{\rho} \frac{\partial p}{\partial z}
\]

\[
\text{(symmetry) \quad \Omega (c6)}
\]

\[
= \frac{-\Delta p}{L} + \mu \left[ \frac{1}{r} + \frac{\partial}{\partial r} \left( \frac{\partial u_r}{\partial r} \right) \right] + \frac{\mu}{\rho} \frac{\partial u_r}{\partial \theta}
\]

\[
\text{(c6)}
\]

\[
+ g z
\]

(mult. both sides by \( r \) before intgrating):

\[
\frac{\partial}{\partial r} \left[ \frac{1}{2} \rho u_r \frac{\partial u_r}{\partial r} \right] = \frac{1}{2} r \frac{\partial}{\partial r} \frac{\partial p}{\partial z} + \frac{\mu}{\rho} \frac{\partial u_r}{\partial \theta}
\]

\[
\frac{\partial u_r}{\partial r} = \frac{1}{2} r \frac{\partial}{\partial r} \frac{\partial p}{\partial z} + \frac{\mu}{\rho} \frac{\partial u_r}{\partial \theta}
\]

\[
\frac{\partial u_z}{\partial r} = \frac{1}{2} \frac{\partial}{\partial r} \frac{\partial p}{\partial z} + \frac{\mu}{\rho} \frac{\partial u_r}{\partial \theta}
\]

\[
\text{At } r = 0 \quad u_r = \frac{\mu}{\rho} \frac{\partial u_r}{\partial \theta}
\]

\[
\text{At } r = R \quad u_r = 0 \text{ (no-slip)}
\]

Thus: \( A = 0 \)

\[
R = -\frac{1}{2} \frac{R^2}{\mu} \frac{\partial p}{\partial z}
\]

or \( u_r = -\frac{1}{2} \frac{R^2}{\mu} \frac{\partial p}{\partial z} \left( 1 - \frac{R^2}{R^2} \right) \)

which is a parabola again!

Gravity would yield the same result, just replace \( \frac{\partial p}{\partial z} \) with \( g \tilde{z} \)!
What is the total flow rate? 
\[ Q = \int_{A} u_z \, dA = \int_{0}^{\alpha} 2\pi u_z \rho \, dr \]

since it’s not a f(x). (8)

Integrating:
\[ Q = 2\pi \left( -\frac{1}{4} \frac{\Delta P}{L} \frac{R^2}{\mu} \right) R^2 \int_{0}^{1} (1 - r^2) r^4 \, dr \]

where \( r^* \approx \frac{r}{R} \)

So:
\[ Q = \frac{\pi}{8} \left( -\frac{\Delta P}{L} \frac{R^4}{\mu} \right) \]

which is known as Poiseuille’s Law & flow thru a tube is also called Poiseuille Flow.

Ok, what is it good for? It is the basis of the capillary viscometer.

\[ \mu = \left( -\frac{\Delta P}{L} \right) \pi \frac{R^4}{Q} \]

Usually, the CP is provided by hydrostatic pressure variation: just make sure it’s not too fast or too slow.

What are the limitations on Poiseuille’s Law? \( \Rightarrow \) Assumption of unidirectional flow!

There are two ways this is violated: entrance effects & turbulence.

Look at turbulence first: \( \Rightarrow \) Flow is too fast, becomes unstable.

Reynolds showed that for a tube, the transition is governed by a dimensionless number:
\[ Re = \frac{UD}{D} \]

Ok, what about entrance length effects? \( \Rightarrow \) Initially, entering flow profile is (more or less) flat.

It must evolve to parabolic shape. How far down the tube does this take?

The flow evolves due to diffusion of momentum, so:
\[ t_0 = \frac{R^2}{\nu} \]

How far does it move during \( t_0 \)?
\[ L \approx t_0 U \sim \frac{U R^2}{\nu} = \frac{1}{4} \frac{U D}{\nu} \]

Actually, the entrance length is usually given as:
\[ L_e = 0.035 \frac{D}{U} \]

which is just a bit numerically smaller!
This is the classical force. It is very important in large scale systems. The most important example is the weather. It's why the world rotates.

We see this in the E component of the equation:

\[ \frac{dE}{dt} = -\frac{\sqrt{2}}{2} \sin^2 \theta + \frac{\sqrt{2}}{2} \cos^2 \theta \]

OK most of these terms are zero due to the coordinate transformations.

Let's look at the E component (where the action is):

\[ \frac{dE}{dt} = \frac{1}{2} \frac{1}{2} m v^2 + \frac{1}{2} \frac{1}{2} m v^2 \]

Thus, if \( v = 1 \) then \( \frac{dE}{dt} = 0 \). Now for the momentum equations:

\[ \frac{dx}{dt} = 0 \]

(No variation in \( \theta \) direction)

Now look at \( x \) and \( r \) components.

Let's look at another problem.

If \( \theta = 90^\circ \) then the local angular velocity is constant.

If \( \theta \) is conserved (say, conservation of kinetic energy), the local rate of rotation is \( \omega = \frac{d\theta}{dt} \).

Now we have \( \omega \) for each body rotation.

To see why this occurs, consider a body undergoing slow radial motion.

\[ P = f_\theta + f_{\phi} \]

which can be expanded.

Now, if \( \phi = 0 \) then \( f_{\phi} = 0 \).

Thus, \( P = f_\theta = 0 \).

Now, if \( \phi = 90^\circ \), then \( f_{\phi} = 0 \).

The centrifugal force term. It is a pseudo-force which arises from the coordinate transformation.

\[ \frac{d^2z}{dt^2} = -\frac{2}{3} \sqrt{2} \sin \frac{1}{2} \theta + \frac{2}{3} \sqrt{2} \sin \frac{1}{2} \theta \]

[Diagram]
On the earth, rotational velocities are much higher than wind velocities, at least on large length scales; thus the Coriolis force is dominant.

\[ \Omega = \frac{2 \pi}{24 \text{ hr}} \approx 4,000 \text{ mi/hr}, \approx 10^5 \text{ mph} \]

On large length scales it’s small (at least due to earth rotation) so the bathtub vortex is due to some initial swirling motion.

Ok, how about Couette flow? \( u_r = 0 \) so Coriolis force doesn’t matter.

\[ \frac{\partial \tau}{\partial \theta} = 0 \quad \text{from symmetry, so:} \]

\[ 0 = \mu \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial \theta} (r u_\theta) \right) \]

We integrate this once:

\[ \frac{1}{r} \frac{\partial}{\partial r} (r u_\theta) = C_1 \]

And a second time:

\[ r u_\theta = \frac{1}{2} C_1 r^2 + C_2 \]

or:

\[ u_\theta = \frac{1}{2} C_1 r^2 + \frac{C_2}{r} \]

We have the no-slip B.C.’s:

\[ u_\theta = \begin{cases} 0 & \text{for } r = R_0 \\ \Omega R & \text{for } r = R_1 \end{cases} \]

Thus:

\[ \frac{1}{2} C_1 R_0^2 + \frac{C_2}{R_0} = 0 \]

\[ \frac{1}{2} C_1 R_1^2 + \frac{C_2}{R_1} = \Omega R_1 \]

So:

\[ C_1 = \frac{-2 C_2}{R_0^2} ; \quad C_2 = -\frac{2 \Omega R_0^2}{R_0^2 - R_1^2} \]

and:

\[ u_\theta = \Omega R \left( \frac{R_0^2}{R_1^2 - R_0^2} \right) \left( \frac{r^2 - R_0^2}{R_0^2} \right) \]

Now \( u_r = 0 \) and \( u_\theta \) is given by:

\[ u_\theta = \frac{\sqrt{2} R^2}{R_1^2 - R_0^2} \left( 1 - \frac{R_0^2}{r^2} \right) \]

So:

\[ \tau_{r\theta} = 2 \mu \frac{\sqrt{2} R^2}{R_1^2 - R_0^2} \frac{R_0^2}{r^2} \]

Now the force \( F \) is just the shear stress \( \tau_{r\theta} \) times the area of the cylinder. Recall \( \tau_{r\theta} \equiv F/A \) exerted by fluid of greater \( r \) on fluid of lesser \( r \) in the \( \theta \) direction.

So:

\[ M = \mu \frac{2 \pi R h}{A_{\text{area}}} \tau_{r\theta} R_0 \hat{\theta} \]

In cylindrical coordinates:

\[ \tau_{r\theta} = \mu \left[ \frac{\partial}{\partial r} (u_\theta) + \frac{1}{r} \frac{\partial}{\partial \theta} (r u_\theta) \right] \]

Now that this is independent of \( n \) this makes sense: the torque exerted by the outer cylinder is the same as that exerted on the inner cylinder, and every cylindrical surface in between. Otherwise the flow would be accelerating (not at steady-state).
OK, what about the thin-gap approximation? Just as the earth looks flat when viewed on a human length scale, so fluid mechanics problems may be simplified when characteristic lengths (e.g. the gap width between cylinders) is much smaller than the radius of curvature!

We take \( R_1 - R_0 \ll 1 \)

Locally, we define coordinates:

\[
\begin{align*}
\mathbf{r} &= r \mathbf{e}_r - z \mathbf{e}_z \\
\mathbf{x} &= \frac{r}{R_1} \mathbf{e}_r - \frac{z}{R_0} \mathbf{e}_z
\end{align*}
\]

The force \( F \) is approximately:

\[ F \approx \tau_{yx} \cdot 2 \pi R_0 h \]

where:

\[
\tau_{yx} \approx \frac{\mu J_2 R_1}{R_1 - R_0}
\]

So:

\[
\left( \frac{\omega}{\omega_0} \right)_{approx} = \frac{J_2 R_1}{R_1 (R_1 - R_0)} \frac{R_1}{R_0}
\]

We can compare this to the exact result:

\[
\left( \frac{\omega}{\omega_0} \right)_{exact} = \frac{1}{2} \frac{R_1^2 - R_0^2}{R_1 (R_1 - R_0)} = 1 - \frac{1}{2} \frac{R_0^2}{R_1}
\]

So if \( R_0 \) is 1" and \( R_1 - R_0 = 0.02" \) (about 500μm), then the error is only around 1%!

In this derivation we have assumed that \( u_2 = u_3 = 0 \). This will be valid provided the rotation rate is sufficiently small. At higher

The critical rotation speed at which vortices appear is given by:

\[
To_{crit} = \frac{J_2 R_1 \pi R_0^3}{3 \mu^2} = 1712 \quad \text{for} \quad \frac{AR}{R} \ll 1
\]

This phenomenon was first demonstrated by G.I. Taylor in 1923.

Note that if \( J_2 \) is further increased, these vortices will themselves become unstable to other secondary flows - they become wavy in the \( \theta \) direction. Eventually, the entire flow becomes turbulent.

Taylor-Couette flow is still actively studied today!
**Dimensional Analysis**

Now that we're familiar with the Navier-Stokes equations, let's use them to look at a more complex general problem: Uniform flow past an arbitrary shape:

\[ \mathbf{u} = U \mathbf{e}_x \]

**Fluid properties**

**P** \( \rightarrow \) **P**₀ as \( |x| \rightarrow \infty \) (far away)

What is the drag (force) on the object?

The force exerted by the fluid on the object is:

\[ F = \int_{S_D} \mathbf{t} \, dA \]

"Mechanical computer" - if the assumptions used in deriving the equations are valid, the experiment should match the solution to the N-S eqns!

To work with a scale model (interpret the results), we have to render the problem dimensionless using appropriate length & time scales.

* All dimensionless variables should be \( O(1) \) in the region of interest!

OK, let's see how this works: We have the Continuity & the N-S eqns:

\[ \nabla \cdot \mathbf{u} = 0 \]  (incompressible)

\[ \nabla \left( \frac{\rho \mathbf{u}}{\rho_0} \right) + \nabla \cdot \left( \mu \nabla \mathbf{u} \right) = -\nabla P + \mu \nabla^2 \mathbf{u} + \rho g \]

(Non-Newtonian Fluid)

Let's choose \( \mathbf{U} \) as the velocity scaling, \( \ell \) as the length scale, \( \ell_U \) as the time scale, and \( \rho_0 \) as the pressure scale (to be determined)

Thus,

\[ u^* = \frac{u}{U}, \quad x^* = \frac{x}{\ell}, \quad v^* = \frac{v}{\ell_U}, \quad P^* = \frac{P - P_0}{\rho_0} \]

\[ g^* = \frac{3}{2} \]  (vector in gravity direction)
Now we rewrite the Navier-Stokes equations:

\[ \frac{\partial}{\partial t} \mathbf{u}^* + \nabla \times \mathbf{u} \times \mathbf{u}^* = -\frac{\partial p^*}{\partial x} \mathbf{e}_x + \frac{\mu^*}{\partial x} \mathbf{e}_x \mathbf{u}^* + \frac{g^*}{U^2} \mathbf{g}^* \]

At high velocities, pressure gradients due to inertial effects (e.g., convection of momentum), so we choose:

\[ \frac{\Delta p^*}{g^*} = 1 \]

or \[ \Delta p^* = 9U^2 \] as the characteristic pressure differential.

Note that we have two dimensionless groups of parameters in the equations. The magnitude of these groups determine the relative importance of the dimensionless terms they multiply and the corresponding physical mechanisms.

What are they?

\[ \frac{\mu^*}{\rho^* U^2} = \frac{1}{Re} = \frac{\text{viscous forces}}{\text{inertial forces}} \]

If \( Re \gg 1 \) then viscous forces are unimportant on a length scale and comparable to the size of the body! We'll see later that they are important in boundary layers next to the body of thickness \( \delta \) because without viscosity, you can't satisfy the no-slip condition.
Sometimes you get additional dimensionless groups from R.C.'s, say if object is rotating. Here there are only two dimensionless groups which contain all the dimensional information! If these are held constant between the model & the full size system, the dimensionless flow will be exactly the same!! This is known as dynamic similarity!

OK, how could we use this? Suppose we want to model a submarine with a 1/100 scale model, preserving dynamic similarity.

the model up to this speed, it still wouldn’t achieve similarity! Our assumptions break down because \( \frac{U_2}{U_0} \neq 1 \) (e.g., the Mach # isn’t small) and so the fluid is compressible.

It can work well, however—suppose we want to look at the flow patterns in a big tank of Karo syrup. We model this with a small tank of water:

\[
U_1 = 25 \text{ strokes} \quad U_2 = 0.01 \text{ strokes} \\
L_1 = 20 \text{ ft} \quad L_2 = 2''
\]

OK, what’s \( U_2 \)?

\[
U_2 = \frac{L_1}{L_2} U_1 \left( \frac{0.01}{25} \right) \left( \frac{240}{2} \right) U_1
\]

If there’s no free surface, it doesn’t matter, so we just have to keep Re. constant. Let \( L_1 \) = length of sub, \( L_2 \) = length of model

For dynamic similarity, \( Re_1 = Re_2 \)

So:

\[
\frac{U_1 L_1}{D_1} = \frac{U_2 L_2}{D_2}
\]

If both experiments are in water, then \( D_1 = D_2 \)

So:

\[
\frac{U_2}{D_2} = \frac{U_1}{D_1}
\]

or

\[
U_2 = U_1 \left( \frac{L_1}{L_2} \right) \left( \frac{0.01}{25} \right) \left( \frac{240}{2} \right)
\]

Note that this is really hard!

If \( U_1 = 40 \text{ mph} \), \( U_2 = 4/00 \text{ mph} \),

Note that even if we could get

\[
U_2 = 0.048 U_1
\]

If \( U_1 = 1 \text{ ft/s} \), \( U_2 = 1.46 \text{ cm/s} \),

which is a reasonable value!

If there is a free surface, we have to preserve both \( Re \) & \( Fr \)!

As an example, consider a vortex in an agitated tank:

To preserve dynamic similarity we require:

\[
Re_1 = Re_2 \quad ; \quad Fr_1 = Fr_2
\]
where \( \text{Re} = \frac{UL}{v} \), \( \text{Fr} = \frac{UL}{g} \)

Note: \( U \approx \text{Fr} \) since all geometric ratios must be preserved as well. So:

\[
\frac{J_1^2 L_1^2}{\rho_1} = \frac{J_2^2 L_2^2}{\rho_2}
\]

\[
\frac{J_2^2 L_2^2}{\rho_2} = \frac{J_2^2 L_2^2}{\rho_2} \frac{3}{9}
\]

Suppose we are modeling a tank of glycerin by one of water. This fixes the ratio \( \rho_1/\rho_2 \)

So:

\[
\frac{J_2^2 L_2^2}{\rho_2} = \frac{J_1^2 L_1^2}{\rho_1} \frac{3}{9} \frac{J_2^2 L_2^2}{\rho_2} = \frac{J_1^2 L_1^2}{\rho_1} \frac{3}{9}
\]

\[
J_2^3 L_2^5 g = f(\text{Re}, \text{Fr})
\]

But if \( \text{Re}, \text{Fr} \) are constant between model system and original, \( f(\text{Re}, \text{Fr}) \)
(unknown \( \text{Re} \) & \( \text{Fr} \) dependence) will also be constant!

Thus:

\[
\frac{(\text{Power})_1}{(\text{Power})_2} = \frac{J_1^3 L_1^5 g}{J_2^2 L_2^5 g_2}
\]

\[
= \left( \frac{J_1}{J_2} \right)^2 \left( \frac{L_1}{L_2} \right)^5 \left( \frac{g_1}{g_2} \right)
\]

\[
= \left( \frac{J_1}{J_2} \right)^2 \left( \frac{L_1}{L_2} \right)^5 \left( \frac{g_1}{g_2} \right)
\]

Which allows us to estimate the power requirements of the full-scale system.

\[
\text{(Power)}_2 = \left( \frac{J_1}{J_2} \right)^2 \left( \frac{L_1}{L_2} \right)^5 \left( \frac{g_1}{g_2} \right)
\]

\[
\frac{(\text{Power})_1}{(\text{Power})_2} = \frac{J_1^3 L_1^5 g}{J_2^2 L_2^5 g_2}
\]

\[
= \left( \frac{J_1}{J_2} \right)^2 \left( \frac{L_1}{L_2} \right)^5 \left( \frac{g_1}{g_2} \right)
\]

\[
= \left( \frac{J_1}{J_2} \right)^2 \left( \frac{L_1}{L_2} \right)^5 \left( \frac{g_1}{g_2} \right)
\]

While strict dynamic similarity is often very difficult (or impossible) to achieve, a more approximate form is much easier and more practical. An excellent example is in hull design for surface ships. Strict similarity requires both \( \text{Re} \) & \( \text{Fr} \) to be preserved between model and full scale, which isn't really possible. If \( \text{Re} \) is high for both ship & model, however, we may be at some "high Re limit" where viscous effects are unimportant. That would mean that only \( \text{Fr} \) would have to be kept constant—much easier!
Let's see how this works:

We wish to model the behavior of the Enterprise (CVN 65) with a 1/100 scale model. In this case $U_1 \approx 40 \text{ mph} = 1,800 \text{ cm/s}$, $L_1 = 1000 \text{ ft} = 30.5 \times 10^4 \text{ cm}$, $\nu = 0.01 \text{ stokes}$

Thus: $Re_1 = 5.4 \times 10^9$, $Fr_1 = 0.11$

we give up on $Re_1$, but try to match $Fr_1$

$$\frac{U_1}{L_1^2} = \frac{U_2}{L_2^2} \Rightarrow U_2 = U_1 \left( \frac{L_1}{L_2} \right)^{\frac{1}{2}}$$

or, since $L_2/L_1 = \frac{1}{100}$,

$$U_2 = \frac{1}{10} U_1 = 4 \text{ mph} = \text{ not bad!}$$

we also have:

$$Re_2 = \frac{U_2}{\nu} = 10^{-3} Re_1 = 5.4 \times 10^6$$

which is still pretty large!

What about the relation between the force on the model and the force on the ship? If viscosity is unimportant we get:

$$\frac{F_1}{8 U_1^2 L_1^2} \approx \frac{F_2}{8 U_2^2 L_2^2}$$

or

$$\frac{F_1}{F_2} \approx \frac{U_2^2 L_2^2}{U_1^2 L_1^2} = 10^6$$

Provided that $Re_2$ is large enough that the flow around the model is fully turbulent ($Re_2 > 10^5$ or so) this actually works pretty well! This has been the basis for testing ship designs over the past century!

So far we've scaled the $\nabla$ equations using the inertial terms (convection of momentum). This is appropriate for $Re >> 1$.

What about low $Re$?? Here we use the viscous scalings!

Recall:

$$\frac{\rho U}{\mu} \left[ \frac{\partial U}{\partial x} + U \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} \right] = -\frac{\partial p}{\partial x} \delta^x$$

$$+ \frac{\mu U}{\rho} \frac{\partial^2 U}{\partial x^2} + g \delta^y$$

This time we divide thru by viscous scaling $\frac{\rho U}{\mu}$:

$$\left( \frac{\rho U}{\mu} \right) \left[ \frac{\partial U}{\partial x} + U \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} \right] = -\frac{\partial p}{\partial x} \delta^x$$

$$+ \frac{\mu U}{\rho} \frac{\partial^2 U}{\partial x^2} + g \delta^y$$

Now we choose $\Delta Re$, e.g.:

$$\frac{\Delta Re}{\Delta U^2} = 1$$

or:

$$\Delta Re = \mu \frac{U}{R}$$

(scaling for shear stress)

Thus:

$$\frac{\partial p}{\partial x} \delta^x - \frac{\partial^2 U}{\partial x^2} + \frac{\partial U}{\partial y} \frac{\partial \partial U}{\partial y} = 0$$

Now if $Re << 1$ we neglect terms which are of $O(Re)$:

$$\nabla \cdot \delta^x = 0 \ (Re, \ Re)$$

or:

$$\delta^x = \delta^y \delta^z$$

and the CE: $\nabla \cdot \delta^x = 0$

These are the Creeping Flow Equils: Starting point for low $Re$ flow!
So far we've used our knowledge of the flow equations to determine conditions where flows will be dynamically similar. This wasn't really necessary =⇒ all that we really had to know was what physical parameters a problem depends on. This is known as dimensional analysis.

The key is that Nature knows no Units: A “foot” or a “meter” has no physical significance. Thus, any physical relationship must be expressible as a relationship between dimensionless quantities!

The dimensional matrix is given by:

\[
\begin{bmatrix}
M & 1 & 0 & 1 & 0 \\
L & 1 & 1 & -3 & 1 \\
T & -2 & 0 & 0 & 1 & -1
\end{bmatrix}
\]

\[
\text{Rank} = \text{Dimension of largest sub-matrix with non-zero determinant!}
\]

In this case, we take the first three columns:

\[
\begin{bmatrix}1 & 0 & 1 \\ 1 & 1 & -3 \\ -2 & 0 & 0 \end{bmatrix} \neq 0 \quad \text{so rank} = 3
\]

By the TT theorem:

\[
\text{Dimensionless groups = 5-3 = 2 and thus} \quad T_1 = h(T_2)
\]

Let’s see how this works—consider drag on a sphere:

\[
U \quad \mu \quad g
\]

The force is a function of \( U, \mu, \rho, g \), but all these are dimensional quantities. How may dimensionless groups can be formed?

Buckingham π theorem:

\[
\text{Dimensionless groups = Parameters - rank of dimensional matrix}
\]

\[
\text{(this is the number of independent fundamental units involved in the problem)}
\]

We may choose \( T_1 \) \& \( T_2 \) any way we wish provided they are 1) dimensionless and 2) independent. (this means that if there are \( N \) π groups, the \( N \)th can’t be formed by any combination of the other \( N-1 \) groups!)

We usually choose groups so that one involves the dependent parameter of interest, and all the others involve combinations of the independent parameters.

One choice:

\[
\frac{F}{\frac{1}{2} U^2 a^2} = f^2 \left( \frac{U \rho g}{a} \right) = f^2(Re)
\]
or, in words, the dimensionless drag is only a function of the Reynolds Number! This is exactly what we got from scaling the N-S Eqns!

Often we can strengthened results if we have additional physical insight. Suppose we have $Re \ll 1$. In this case we expect inertia (and hence $g$) is unimportant:

\[
\frac{F}{\mu U} = \frac{f^2}{U \mu a} \quad \frac{1}{4 \pi} \quad 0 \quad 0
\]

Thus:

\[
\text{rank} = 3
\]

\[\therefore \ 4 - 3 = 1 \quad \text{group}
\]

Again, there are $5 - 3 = 2$ dimensionless groups:

\[
\frac{F_L}{\frac{g U^3 a}{\mu}} = \frac{f^2}{U \mu a} \quad \text{(so this works fine!)}
\]

What about flow near $Re = 1$? We anticipate that $g$ (inertia) doesn't matter, so we have:

\[
\frac{F_L}{\frac{g U^3 a}{\mu}} = \frac{f^2}{U \mu a} \quad \text{(or \ 4 - 3 = 1 \ dimensional groups.)}
\]

Thus:

\[
\frac{F_L}{\frac{g U^3 a}{\mu}} = \text{cst}
\]

But this suggests that the drag is independent of $a$! This can't be correct! This reflects the fact that inertia is always important: there is no solution to the Stokes Eqns for 2-D flow past a cylinder! This is known as the 

Stokes' Paradox:

An approximate solution for $Re \ll 1$ is given by Lamb:

\[
\frac{F_L}{\frac{g U^3 a}{\mu}} \approx 4 \pi \frac{U \mu}{\ln \left( \frac{4 a}{U} \right) - \frac{\pi}{2}}
\]

Buler's Coast

Which depends on $Re$ even as $Re \to 0$!

The complete reduction of a problem to a single dimensionless group sometimes happens even for functions.

The best example of this is the
Expanding shock wave due to a point source explosion studied by GI Taylor during WWII. The radius $R$ of the shock will be a function of time $t$, the density of the gas $(\rho_0)$, the energy $E$, the adiabatic exponent $\gamma = 7/5$ for a diatomic gas, and the initial atmospheric pressure $P_0$. Thus:

$$R = f^a (t; \rho_0, E, P_0, \gamma)$$

$$M = 0 \cdot -3 \cdot 1 \cdot 1 \cdot 0$$

$$T = 1 \cdot 0 \cdot -2 \cdot 2 \cdot 0$$

Thus 6-3 = 3 groups!
One is obviously $\gamma$, but this won’t change if we keep using air!

Thus we have:

$$R = f^a (t; \rho_0, E, \gamma)$$

or 8-3 = 2 groups!
Since one is still $\gamma$, the other is:

$$R = \left( \frac{E^{2/5}}{\rho_0^{2/5}} \right)^{1/5} = f (\gamma) = \text{cst} \text{ for diatomic gases}.$$  

It turns out that this constant is 1.033 from solution of the flow equations! Thus $R \propto t^{2/5}$ and with knowledge of $R$ & $t$ you can calculate $E$. This was done by Taylor from images of the NM atom bomb tests - while the yields were still classified Top Secret!

We can construct a reference length and time:

$$\frac{R}{(E/P_0)^{1/5}} = f^b \left( \frac{t}{\left( \frac{E_{25}}{P_0^2} \right)^{1/5}} \right)^{1/5}$$

Which isn’t particularly useful. Still, we could plug this into the shock eq’ns & try to solve the problem. Instead we look at the case of strong explosions such that $P_0 R^2 < E$

In this case the pressure inside the shock is far greater than that due to the atmosphere $P_0$. Thus, we shall assume that $P_0$ doesn’t matter!

As a last point, while fundamental units *are* fundamental units, this isn’t always true. As an example, consider the deflection produced by a ball hitting an elastic solid (e.g., a ball bearing on a block of rubber):

$$F = \frac{h}{k}$$

where $h = f^a (F, a, E)$

These involve $M, L, T$, so we might expect 4-3 = 1 dimensionless groups! Thus $\frac{h}{k} = \text{cst} \ ??$
This can’t be correct, since elasticity \( E \) must matter!
The problem is in the rank of the dimensional matrix:
\[
\mathbf{h} = \mathbf{f}^d(F, a, E)
\]
\[
\begin{pmatrix}
1 & 0 & 1 \\
1 & -2 & 0 \\
0 & 1 & -2
\end{pmatrix}
\]
There exists no 3x3 matrix with non-zero determinant, thus rank = 2
So:
\[
\frac{h}{a} = \mathbf{f}^d \left( \frac{F}{\varepsilon a^2} \right)
\]
which makes more sense!

Dimensional Analysis is powerful, but be careful and always check to see if your results make sense!

Lubrication Flows

An important problem in fluid mechanics is lubrication theory: the study of the flow in thin films, where hydrodynamic forces keep solid surfaces out of contact, reducing wear. These problems are actually quite simple to solve due to a separation of length scales (one dimension >> another) which leads to the quasi-parallel flow approximation.

Let’s see how this works.
Suppose we look at the squeeze flow between a disk and a plane:

We also take:
\[
\dot{\psi} = \frac{\dot{u}_r}{R}, \quad \dot{z}^* = \frac{z}{H}
\]
So:
\[
\frac{U}{R} \frac{1}{\mu \delta} \left( \frac{\dot{u}_r}{R} \right) + \frac{U}{R} \frac{2 \dot{u}_z}{\delta H} = 0
\]
or, dividing through:
\[
\frac{1}{\mu \delta} \frac{2 \dot{u}_z}{R} + \frac{RV}{UH} \frac{2 \dot{u}_z}{\delta H} = 0
\]
Both terms of the CE must be of the same order for any 2-D problem. Thus we take:
\[
\frac{RV}{UH} = 1 \quad \text{or} \quad U = \frac{R}{H} \quad V \gg V
\]
Thus the velocity along the gap is much higher (by \( O(H^2) \)) than the velocity perpendicular to the gap. This means we have quasi-parallel flow in the radial direction.
Now for the momentum equations:

\[ \text{r'-momentum:} \]

\[ g \left( \frac{\partial u_r^*}{\partial z^*} + u_r^* \frac{\partial u_r^*}{\partial r^*} + u_z^* \frac{\partial u_r^*}{\partial z^*} \right) = -\frac{2p^*}{\partial r^*} + \mu \left[ \frac{\partial}{\partial r^*} \left( \frac{1}{r^*} + \frac{\mu}{\partial r^*} (\mu u_r^*) + \frac{\partial u_r^*}{\partial z^*} \right) \right] \]

where we have ignored \( \Theta \) terms.

Scaling:

\[ \frac{H^2}{K} \left( \frac{\partial u_r^*}{\partial z^*} + u_r^* \frac{\partial u_r^*}{\partial r^*} + u_z^* \frac{\partial u_r^*}{\partial z^*} \right) = -\frac{\partial p^*}{\partial r^*} + \mu \left[ \frac{\partial}{\partial r^*} \left( \frac{1}{r^*} + \frac{\mu}{\partial r^*} (\mu u_r^*) + \frac{\partial u_r^*}{\partial z^*} \right) \right] \]

In lubrication flows we expect viscous terms to dominate, so

Provided that \( \frac{H^2}{K} \ll 1 \) we can neglect the \( r^* \)-diffusion terms, and provided \( \frac{2qH}{\mu} \ll 1 \) we can ignore the inertial terms.

Thus:

\[ \frac{\partial u_r^*}{\partial z^*} = \frac{2p^*}{\partial r^*} \]

which is just channel flow! (\( r^* \) \( u^* \) \( z^* \))

with boundary conditions:

\[ u_r^* \big|_{z^*=0,1} = 0 \]

we get:

\[ u_r^* = \frac{\partial p^*}{\partial r^*} \frac{1}{2} (1 - z^*)^2 \]

Now we still need to figure out the pressure gradient. We do this from a mass balance.

\[ \text{Flow out thru top} = V \pi r^2 \]

\[ \text{Flow in thru sides} = 2\pi r \int_0^1 u_r^* \, dz \]

These must balance!

\[ V \pi r^2 = -2\pi r \int_0^1 u_r^* \, dz \]

or

\[ \int_0^1 u_r^* \, dz = -\frac{1}{2} r^* \]

So:

\[ \int_0^1 \left( \frac{\partial p^*}{\partial r^*} \right) (1 - z^*)^2 \, dz^* = -\frac{1}{2} r^* \]

\[ \left. \frac{\partial p^*}{\partial r^*} \right|_{z^*=1} = \frac{\gamma}{2} \left( \frac{1}{2} r^* \right)^2 \]

Now since \( p^* \big|_{z^*=0} = 0 \), we get:

\[ p^* = -3(1 - r^*) \]
The force is just the integral of the pressure (normal force)
\[ F = \int_0^R p \, 2\pi r \, dr = 2\pi R^2 \int_0^R \left( \frac{2\pi R^2}{H^2} \right) r \, dr \]
\[ = -\frac{3\pi}{2} \left( \frac{R^2}{H^2} \right) R^2 \]

or, since \( U = V \frac{R}{H} \),
\[ F = -\frac{3\pi}{2} \frac{MV}{H} \]

Note, the force blows up as \( H \to 0 \). This is characteristic of lubrication flows!

How long does it take to detach from the plane? It spends all the time travelling the first little bit!

For a constant force \( F \),

An important problem in lubrication theory is the sliding block:

If \( H \ll L \) we can use lubrication theory to calculate the upward force on the block for some \( U, a_1, a_2, L, \) etc.

We have the equations:
\[ \delta \left( \frac{2\pi R^2}{H^2} + \frac{\partial^2}{\partial x^2} \right) u = -\frac{2\pi R^2}{H^2} + \frac{\partial^2}{\partial y^2} u \]
\[ \frac{\partial u}{\partial y} = 0 \]

The flow is two-dimensional, so we take \( U = u_x, \ V = u_y \) and \( u_z = \frac{\partial u}{\partial z} = 0 \) (no \( z \)-dependence)

we have the C.E.:
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

As before, we scale \( u \) with \( U \), \( x \) with \( L \), and \( y \) with \( H \):
\[ x^* = \frac{x}{L}, \ y^* = \frac{y}{H}, \ u^* = \frac{u}{U} \]

where all \( \ast \) variables are \( O(1) \) in the region of interest. To preserve both terms in the C.E., we require:
\[ V^* = \frac{V}{U \frac{H}{L}} \]

Thus \( \frac{2u^*}{\partial x^*} + \frac{2v^*}{\partial y^*} = 0 \)

We shall define \( \varepsilon = \frac{H}{L} \ll 1 \)

Thus \( V \) is \( O(\varepsilon U) \).

OK, now for \( x \)-momentum:
Let \( t^* = \frac{Ut}{L} \) (e.g., \( t_0 = \frac{L}{U} \)) and \( p^* = \frac{P- P_0}{\rho_0} \). Plugging in,
\[
9 \frac{U^2}{L} \left( \frac{\partial u^*}{\partial x} + u^* \frac{\partial u^*}{\partial x} + v^* \frac{\partial u^*}{\partial y} \right) = - \frac{\partial p^*}{\partial x} + \mu \left( \frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2} \right)
\]

We anticipate that the dominant mechanism for momentum transport is viscous shear stresses in the narrow gap. Thus we divide by \( \frac{U^2}{H} \), it's scaling!

Note that we can determine the scale of the force on the block with no further work! The upward force is just:
\[
F = \int_0^L \left( F - P_0 \right) dx
\]

or
\[
F = \frac{U^2 H w}{H} \cdot \text{cst}
\]

where to get the cst we have to solve the problem!

Now for the y-momentum equation:
\[
9 \left( \frac{\partial v^*}{\partial t} + u^* \frac{\partial v^*}{\partial x} + v^* \frac{\partial v^*}{\partial y} \right) = \frac{\partial p^*}{\partial y} + \mu \left( \frac{\partial^2 v^*}{\partial x^2} + \frac{\partial^2 v^*}{\partial y^2} \right)
\]

Thus we have the pressure scale
\[
\Delta p_c = \frac{\mu L}{H^2}
\]

and:
\[
\frac{\partial^2 u^*}{\partial y^2} - \varepsilon \frac{\partial^2 u^*}{\partial x^2} + \varepsilon^2 \frac{\partial^2 u^*}{\partial x^2} + \varepsilon^3 \text{Re} \left( \frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} \right)
\]

We shall ignore terms of \( O(\varepsilon^2) \) and \( O(\varepsilon^3 \text{Re}) \). Thus:
\[
\frac{\partial^2 u^*}{\partial y^2} = \frac{\partial p^*}{\partial x}
\]

Plugging in our scalings we get:
\[
\frac{\partial p^*}{\partial y} = \varepsilon^2 \frac{\partial u^*}{\partial x} + \varepsilon^4 \frac{\partial^2 u^*}{\partial x^2}
\]

Thus, if we ignore terms of \( O(\varepsilon^2, \varepsilon^3, \varepsilon^3 \text{Re}) \) we get:
\[
\frac{\partial p^*}{\partial y} = 0
\]

This is generically true for problems with separations of length scales \( H/L \ll 1 \), which also occurs in boundary layer flows we'll study later. Basically, you don't get variations in pressure across the thin dimension, in this case the gap!
OK, we have the scaled eqns:
\[ \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0 \]
\[ \frac{\partial^2 \tilde{u}}{\partial y^2} = \frac{\partial \tilde{p}^*}{\partial x} \]
\[ \frac{\partial \tilde{p}^*}{\partial y} = 0 \]

Now for the B.C.'s:
\[ \tilde{u}^* \bigg|_{y=0} = \tilde{v}^* \bigg|_{y=0} = 0 \] (is stationary)
and for the moving surface:
\[ \tilde{h} = \frac{h}{H} \]

Let \[ \tilde{u}^* \bigg|_{y=h} = U^* = \frac{U(x,t)}{U} \]

In our case \( U^* = 1 \) (uniform velocity)
but in general it could be a function of both \( x \) & \( t \).

Likewise:
\[ \tilde{v}^* \bigg|_{y=h} = \frac{V(x,t)}{U} \]

We still need an equation for \( \tilde{u}^* \).
To get it we look at the C.E.:
\[ \frac{\partial \tilde{v}^*}{\partial y} = -\frac{\partial \tilde{u}^*}{\partial x} \]

We can integrate this to get \( \tilde{v}^* \):
\[ \tilde{v}^* = \int \left( -\frac{\partial \tilde{u}^*}{\partial x} \right) dy^* \]

The lower limit is zero to satisfy the B.C.:
\[ \tilde{v}^* \bigg|_{y=0} = 0 \]

We can evaluate this at \( y = h^* \):
\[ \tilde{v}^* \bigg|_{y=h^*} = \tilde{v}^* \bigg|_{y=0} = \int_0^{h^*} \left( -\frac{\partial \tilde{u}^*}{\partial x} \right) dy^* \]

This gives us an equation for the pressure gradient!

For our example problem \( V = \beta \)

To solve this problem we integrate the x-momentum eqn over \( y \)!
We can do this because \( \tilde{p}^* \) isn't a function of \( y \!
\]
(e.g., \( \frac{\partial \tilde{p}^*}{\partial y} = 0 \))

So:
\[ \tilde{u}^* = \frac{1}{2} \left( \frac{\partial \tilde{p}^*}{\partial x} \right) y^* + C_1(x,t) y^* + C_2(x,t) \]

If we apply B.C. at \( y^* = 0 \) we get \( C_2(x,t) = 0 \)

Applying B.C. at \( y^* = h^* \) gives:
\[ \tilde{u}^* = \frac{1}{2} \left( \frac{\partial \tilde{p}^*}{\partial x} \right) y^* \left( y^* - h^* \right) + U^* \frac{y^*}{h^*} \]

Channel Flow

So \( \tilde{u}^* \) is just the sum of channel & shear flows.

\[ V^* = \frac{1}{12} \left( \frac{\partial \tilde{p}^*}{\partial x} \right) h^* + \frac{1}{4} \frac{\partial \tilde{p}^*}{\partial x} h^* + \frac{\partial \tilde{h}^*}{\partial x} \]

This is known as the Reynolds Lubrication Equation. Together with the B.C.'s \( \tilde{p}^* \bigg|_{x=0} = \tilde{p}^* \bigg|_{x=h} \)
we can calculate the pressure!

OK, let's apply this:
\[ \tilde{p}^* \bigg|_{x=\theta_1, \theta_2} = \frac{\partial \tilde{h}^*}{\partial x} \]

\[ H = \theta_1, \quad h = \frac{\theta_2 - \theta_1}{L} x + \theta_1 \]
In dimensionless form,
\[ h^* = 1 + \frac{\alpha - \alpha_1}{\alpha_1} x^* \]
we also have:
\[ \text{let } \frac{\alpha - \alpha_1}{\alpha_1} \text{ be } \Delta \] 
\[ U^* = 1, \quad V^* = 0 \]
\[ \frac{\partial}{\partial x^*} \left( h^* \frac{\partial h^*}{\partial x^*} \right) = \frac{3}{2} \frac{\partial \alpha}{\partial x^*} \]
Integrating once:
\[ h^* \frac{\partial h^*}{\partial x^*} = - \frac{3}{2} \frac{\partial \alpha}{\partial x^*} x^* + C_1 \]
Dividing by \( h^* \frac{3}{2} \) and integrating again:
\[ p^* = - \frac{3}{2} \frac{\partial \alpha}{\partial x^*} x^* + C_1 \int \frac{\partial h^*}{\partial x^*} dx^* \]
where the second constant of
integration vanishes because
\[ p^* \big|_{x^*=0} = 0. \]
Evaluating this at \( x^* = 1 \) and applying the \( p^* \big|_{x^*=1} = 0 \) B.C. yields:
\[ C_1 = 6 \frac{\partial \alpha}{\partial x^*} \frac{1}{h^*} \int_0^1 \frac{dx^*}{h^*} = \frac{6 \frac{\partial \alpha}{\partial x^*}}{h^*} \]

So:
\[ p^* = \frac{6 \frac{\partial \alpha}{\partial x^*}}{h^*} \frac{x^* - x^*}{(1 + \frac{\alpha - \alpha_1}{\alpha_1} x^*)} \]
The force is just the integral of this:
\[ F^* = \frac{F}{U M L W} = \int_0^1 p^* dx^* \]
\[ = \frac{6 \frac{\partial \alpha}{\partial x^*}}{h^*} \left[ \frac{(\Delta \alpha) \ln(1 + \frac{\alpha - \alpha_1}{\alpha_1})}{\frac{\partial \alpha}{\partial x^*}} \right] \]

The Stream Function

Lubrication flows were an example of quasi-parallel flows: flows where the characteristic length scales were sufficiently different such that the 2-D flow was essentially 1-D. If the length scales are not different, a 2-D flow remains 2-D & we must use a different approach!

Suppose we have an incomp. 2-D flow:
we have the O.E.:
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

If we define the scalar function \( \psi(x, y) \) by:
\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \]

this has the property that the O.E. is satisfied automatically:
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0 \]

Basically, by doing this stream function substitution we are reducing the number of dependent variables (e.g., \( u, v \) to \( \psi \)) while increasing the order of the differential equation.

The stream function has many useful properties! First, it is constant along a streamline.

Remember the material derivative?
\[ \frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + u \cdot \nabla \psi \]

The \( \frac{\partial \psi}{\partial t} \) term is the change in the direction of motion! For the stream \( \psi \), \( \frac{\partial \psi}{\partial t} = 0 \)

We can prove this:
\[ u \cdot \nabla \psi = u \frac{\partial \psi}{\partial x} + v \frac{\partial \psi}{\partial y} \]
\[ = \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial y} \right) = 0 \]

So curves of constant \( \psi \) are streamlines: they follow the motion of fluid elements! That's why \( \psi \) is called the stream function!

Another property: suppose we want to calculate the flow rate through any segment of the flow:

For a unit normal:
\[ n = (n_x, n_y) \]

the tangent is \((-n_y, n_x)\)

So:
\[ \oint_C \mathbf{v} \cdot d\mathbf{s} = \oint_C (u, v) \cdot (n_x, n_y) \, ds \]
\[ = \oint_C \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) \cdot (n_x, n_y) \, ds \]
\[ = \oint_C (-v_x, u_x) \cdot (\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}) \, ds \]
\[ = \oint_C \nabla \psi \cdot d\mathbf{s} \equiv \psi(B) - \psi(A) \]

So the flow rate through any arc from A to B is just the difference in the stream function at these points!
Okay, how do you get $\frac{\partial^2 \eta}{\partial t^2}$? Let's plug into the N-S eqns:

1. $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] = \frac{\partial^2 u}{\partial t^2}$
2. $\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} + \mu \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] = \frac{\partial^2 v}{\partial t^2}$

Let's just look at the RHS of these eqns:

$\text{RHS}_1 = \frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 x}{\partial x^2} + \frac{\partial^2 x}{\partial y^2} \right] + 8 q x$

$\text{RHS}_2 = \frac{\partial p}{\partial y} + \mu \left[ \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial y^2} \right] + 8 q y$

We can eliminate the $p$ terms by

Because the LHS is so nasty, we usually use this eqn only for $Re << 1$ when we can ignore the LHS!

For low $Re$, we have the Biharmonic Equation:

$$\nabla^4 \eta = 0$$

with appropriate B.C.'s

This equation appears in other physical problems too - particularly in the deflection of thin elastic plates! The streamfunction is identical to the deflection of an elastic plate (like a thin sheet of glass) with the same B.C.'s such as the value of $\eta$ or its derivatives on the boundary. This provides a good way of visualizing the spatial dependence of $\eta$ - just visualize the corresponding deflection problem!

Okay, let's work an example. Suppose we have the wiper scraping fluid off a plate. What does the flow look like?

We have $\nabla^4 \eta = 0$
We'll use cylindrical coordinates:

\[ u_\theta = \frac{\partial u}{\partial \theta} \]

Now in cylindrical coords, we have:

\[ \nabla^2 u = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial z^2} \]

\[ \nabla^2 u = \nabla^2 (\nabla^2 u) \]

and B.C.'s:

\[ u = u_\theta = 0 \quad \theta = 0 \]

\[ u = u_\theta = 0 \quad \theta = \theta_0 \]

In terms of \( \psi \) these become:

\[ \frac{\partial \psi}{\partial \rho} = -u_\rho \quad \frac{\partial \psi}{\partial \theta} = 0 \quad \rho = \rho_0 \]

\[ \frac{\partial \psi}{\partial \theta} = \frac{\partial \psi}{\partial z} = 0 \quad \theta = \theta_0 \]

This has the general solution:

\[ \psi = A \sin \theta + B \cos \theta + C \sin \theta \cos \theta + D \cos \theta \sin \theta \]

where the constants are left.

From the B.C.'s:

\[ f(0) = -1, \quad f(\theta) = 0 \quad f'(\theta) = 0 \]

Now from \( f(\theta) = 0 \) we get \( B = 0 \)

After some algebra:

\[ f(\theta) = -1 - \sin^2 \theta \left[ \frac{\partial^2}{\partial \theta^2} \sin \theta \right] \]

OK, what is the pressure distribution in the fluid (and on the wiper)?

The inhomogeneous B.C. suggests a solution of the form:

\[ y = u \psi f(\theta) \]

where \( f(\theta) \) has the B.C.'s:

\[ f(0) = -1, \quad f(\theta) = 0 \]

\[ f'(\theta) = f'(0) = 0 \]

Let's see if this works:

\[ \nabla^2 y = \nabla^2 (u \psi f(\theta)) \]

\[ = \frac{\nabla^2 u}{\nabla^2} \left[ \frac{\nabla^2 u}{\nabla^2} (f + f') \right] \]

\[ \frac{\nabla^2 u}{\nabla^2} (f + f') - \frac{\nabla^2}{\nabla^2} (f \psi + \theta \psi f') = 0 \]

This reduces to:

\[ f'' + 2f' + f = 0 \]

In cylindrical coords, we have the N-S eqns (RHS only!):

\[ \frac{\partial P}{\partial \theta} = \nu \left[ \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} (u \psi f) \right) \right] \]

\[ + \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} (u \psi f) \right) - \frac{2}{\rho^2} \frac{\partial}{\partial \theta} (u \psi f) \]

Recall:

\[ u_\theta = -\frac{\partial P}{\partial \rho} = -u_\psi \]

\[ u_\rho = \frac{1}{\rho} \frac{\partial P}{\partial \theta} = u_\phi' \]

So:

\[ \frac{\partial^2 P}{\partial \rho^2} = \frac{\mu}{\rho^2} \left[ -f'' + f'' + 2f' \right] \]

\[ = \frac{\mu}{\rho^2} [f' + f''] \]

Thus \( p \sim \frac{\mu}{\rho} \)
Note that this is singular (blows up) as \( r \to 0 \)! This isn’t even an integrable singularity as the total force on the wiper diverges as \( \log(r) \) as \( r \to 0 \)!

Basically, this huge force pushes the wiper off the surface, leaving a thin film behind!

A classic stream-function problem is creeping flow (\( \text{Re} < 1 \)) past a sphere.

Suppose a sphere of radius \( a \) is moving uy velocity \( U \) in the \( z \)-direction. The flow is fully 2-D, but it is axisymmetric.

We choose a spherical coord system such as that given below:

![Spherical coordinates diagram]

Basically, \( \theta \) is the latitude & \( \phi \) is the longitude!

Thus:
\[
\begin{align*}
\epsilon &= r \cos \theta \\
x &= r \sin \theta \cos \phi \\
y &= r \sin \theta \sin \phi
\end{align*}
\]

For this problem, \( u_\theta = 0 \) and there is no \( \phi \) dependence!

we have the B.C.’s:
\[
\begin{align*}
| u_r | &= U \cos \theta; \quad | u_\theta | = -U \sin \theta \\
& \text{as } r = a \\
u_r, u_\theta \to 0 \text{ as } r \to \infty
\end{align*}
\]

In spherical coordinates we have the C.E.:
\[
E^2 \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{r^2 \sin \theta} \frac{\partial^2 \psi}{\partial \phi^2} = 0
\]

The structure of the C.E. suggests:
\[
\epsilon_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}
\]
\[
\epsilon_\theta = \frac{1}{r^2} \frac{\partial \psi}{\partial \phi}
\]

which automatically satisfies the C.E.!

This is not the same as \( \psi \) for 2-D flow & leads to a different equation! For axisymmetric flows at \( \text{Re} = 0 \):

\[
E^4 \psi = 0
\]
where:
\[
E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right)
\]
Ok, how do we solve this? We look at the B.C.'s:

\[ u_r \bigg|_{r=0} = \frac{2\pi}{\sin \theta} \frac{\partial^2 u}{\partial r^2} \bigg|_{r=0} = -U \sin \theta \]

thus \[ \frac{\partial^2 u}{\partial r^2} \bigg|_{r=0} = -U \sin \theta \]

and \[ u_r \bigg|_{r=0} = \frac{1}{\mu^2 \sin \theta} \frac{\partial u}{\partial r} \bigg|_{r=0} = U \cos \theta \]

Thus: \[ \frac{\partial u}{\partial r} \bigg|_{r=0} = -U \sin \theta \cos \theta \]

The structure of these B.C.'s suggests a solution of the form:

\[ u = \sin^2 \theta f(r) \]

We'll try this and see if it works.

Now we have to derive a B.E. for \( f(r) \):

\[ E^2 u = E^2 (E^2 f) = E^2 (E^2 (\sin^2 \theta f(r))) \]

Recall:

\[ E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) \]

\[ = \sin^2 \theta \left( f'' + \frac{f'}{r^2} \right) \]

Similarly, \[ E^2 f = \left( f'' + \frac{f'}{r^2} - \frac{f}{r^4} \right) \sin^2 \theta \]

Thus we get the 4th order ODE:

\[ r^4 f'''' - 4r^3 f''' + 8r^2 f'' - 8r f' + 8r f = 0 \]

\% B.C.'s: \( f(a) = \pm Ua^2, f'(a) = -Ua \)

Plug into B.C.'s:

\[ \frac{\partial u}{\partial r} \bigg|_{r=0} = 2 \sin \theta \cos \theta \]

Thus \[ f(a) = -\frac{1}{2} Ua^2 \]

\[ \frac{\partial u}{\partial r} \bigg|_{r=0} = \sin^2 \theta f(a) = -Ua \sin \theta \]

Thus \[ f'(a) = -Ua \]

So far, so good! Now for the B.C.'s at \( r = \infty \):

\[ u_r \bigg|_{r=\infty} = 0 = \sin \theta \frac{f(r)}{r^2} \bigg|_{r=\infty} \]

so \[ \lim_{r \to \infty} \frac{f(r)}{r^2} = 0 \]

and \[ u_r \bigg|_{r=\infty} = 0 = -\cos \theta \frac{f(r)}{r^2} \bigg|_{r=\infty} \]

so \[ \lim_{r \to \infty} \frac{f(r)}{r^2} = 0 \]

\[ \lim_{r \to \infty} \frac{f(r)}{r^2} = 0, \lim_{r \to \infty} \frac{f(r)}{r^4} = 0 \]

Now since all the terms in the ODE have the form \( r^m \), the general solution is of the form:

\[ f(r) = r^n \]

Plugging in yields the polynomial:

\[ n(n-1)(n-2)(n-3) + 8n-8 = 0 \]

This has 4 roots:

\[ n = -1, 1, 2, 4 \]

Thus:

\[ f(r) = \frac{a}{r} + br + cr^2 + dr^4 \]

where the constants are determined from the B.C.'s!

The condition that \( f(r) \) die off at \( r = \infty \) requires \( c = d = 0 \)
Thus:
\[ f(v) = \frac{\xi}{v} + bv \]

Now at \( r = a \):
\[ f(a) = \frac{\xi}{a} + ba = \frac{1}{2} Ua^2 \]
and
\[ f'(a) = -\frac{\xi}{a} + b = -Ua \]

Solving for \( \xi \) and \( b \) we get:
\[ \xi = \frac{1}{4} Ua^3, \quad b = \frac{3}{4} Ua \]
\[ f = Ua^2 \left( \frac{1}{4} \frac{\xi}{Ua} - \frac{3}{4} \frac{b}{Ua} \right) \cos \theta \]
which gives the velocities:
\[ u_r = -U \cos \theta \left\{ \frac{(\xi)}{2} \right\} - 3 \frac{b}{Ua} \]
\[ u_\theta = -U \sin \theta \left( \frac{(\xi)}{4} \right) - 3 \frac{b}{Ua} \]

We can also obtain the pressure distribution:

\[ \text{on the sphere!} \]
\[ F = \int_0^a \rho U A \delta \]

Recall that:
\[ \sigma = -p \n + \frac{\xi}{2} \]

\[ \text{isotropic part} \]

Thus:
\[ F = \int_0^a p \delta A + \int_0^a \frac{\xi}{2} \cdot 2 \pi \frac{a^2}{2} \]

\[ \text{normal forces} \quad \text{shear forces} \]

\( \text{Form Drag} \quad \text{Skin Friction} \)

At high \( Re \), form drag is large, while skin friction is negligible!
At low \( Re \), both are comparable!

\[ P = P_0 + \frac{3}{2} \frac{\mu a U \cos \theta}{\nu^2} \]

It's important to note that the velocity dies off only as \( O(\frac{r}{R}) \)
for large \( r \). This means that as \( Re \to 0 \), the disturbance produced
by a sphere is felt at very large distances! You have to go \( \sim 100 
\]

radii for the velocity to drop to 1% of the value at the sphere. This means
that boundaries have a strong influence on the motion of objects — an
important result in low \( Re \) flows.

OK, now we have the velocity and the pressure. What about the
drag? (force exerted by the fluid)

The integrals are a bit messy to evaluate, but eventually you get:
\[ F = -\frac{\xi}{2} \int_0^a 2 \pi \mu a U + 4 \pi \mu a U \]

\[ \text{Form Drag} \quad \text{Skin Friction} \]

\[ = -G \pi \mu a U \]

This is known as \( \text{Stokes' Law} \)
and is of fundamental importance in the study of suspensions at
low \( Re \). You should remember this!!

Note that from pure dimensional analysis we had:
\[ \frac{F}{\mu a U} = C_1 \text{ for } Re = 1 \]

Getting the value of the constant
took all the effort!
There's an alternative way to calculate the drag: by an energy balance.

Since there's no accumulation of momentum (kinetic energy), all of the work done by the sphere on the fluid is dissipated in heat! The work done by the sphere on the fluid is just:

\[
\text{Work} = \frac{\text{Force on fluid}}{\text{Time}} = \frac{\text{Total viscous dissipation}}{\text{Total volume}}
\]

The viscous dissipation per unit volume is \( \frac{\partial U}{\partial x} \).

Or in index notation:

\[
\frac{\partial U_i}{\partial x_i}
\]

Thus:

\[
F \cdot U = \int_0^\infty \frac{\partial U}{\partial x} \cdot 2 \pi r \, dr
\]

This yields the same result!

Before we leave creeping flow (e.g., \( Re \ll 1 \)) let's look at another special property: Minimum Dissipation Theorem. Proving this is beyond this course (it's covered in 544), but we can use the result!

Among the set of all vector fields \( \mathbf{U} \) which satisfy:

1. the no-slip conditions on a body (e.g., \( \mathbf{U} \big|_{\partial D} = \mathbf{U}(x) \))

and

2. satisfy \( \nabla \cdot \mathbf{U} = 0 \) (continuity)

then the velocity field which also satisfies the creeping flow equations results in the minimum viscous dissipation!

Since dissipation \( \equiv \mathbf{F} \cdot \mathbf{U} \), this provides a means of estimating the drag on a complex shape!

Example: what is the drag on a cube with sides of length \( s \)?

A corollary to the minimum dissipation theorem is that the drag on any object is less than that on one which completely encloses it! This is only true for \( Re \ll 1 \).

OK, how about the cube?

It's drag is greater than that of a sphere of radius \( \frac{s}{2} \) (which it encloses) but less than that of a sphere of radius \( s \sqrt{3} \), which encloses it!

\[
F_{\text{cube}} < \frac{s}{2} \sqrt{6 \pi U} < \frac{s}{2} \sqrt{6 \pi \mu U s \sqrt{3}}
\]
These are rigorous bounds provided Re \( < 1 \) (higher Re is very different!). We can also estimate the drag by just taking the geometric mean:

\[
F_{	ext{tube}} = 6 T M U S \left( \frac{3}{2} \right)^{3/4}
\]

Another consequence of the minimum dissipation theorem is that streamlining an object by enclosing it in a smooth shell only increases the drag! This is certainly not true for higher Re!

We can eliminate the \( g \) term by defining an augmented pressure:

\[
P = p - g \mathbf{g} \cdot \mathbf{x}
\]

Thus:

\[
\nabla P = \nabla p - g \mathbf{g}
\]

provided \( g \) is constant.

So:

\[
\frac{\partial P}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P
\]

We have the vector identity:

\[
\nabla \cdot \mathbf{u} = \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u})
\]

\[
\nabla \left( \frac{1}{2} \mathbf{u}^2 \right) \quad \text{(vorticity)}
\]

Thus:

\[
\frac{\partial P}{\partial t} + \mathbf{u} \cdot \nabla \left( \frac{1}{2} \mathbf{u}^2 \right) + \nabla P = \mathbf{u} \times \mathbf{\omega}
\]

or

\[
\frac{\partial P}{\partial t} + \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{u}^2 + P - g \mathbf{g} \cdot \mathbf{x}) = \mathbf{u} \times \mathbf{\omega}
\]

Ok, we’ve looked at low Re flows. Now let’s look at high Re limit.

Recall the high Re scaling:

\[
x = \frac{x^*}{U}, \quad y = \frac{y^*}{U}, \quad t = \frac{t^*}{U}
\]

\[
\mathbf{p} = \frac{P - P^*}{(3/2) U^2} \quad \text{inertial scaling}
\]

Thus:

\[
\frac{\partial}{\partial t} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) - \nabla \cdot \mathbf{u}^* + \frac{1}{Re} \nabla^2 \mathbf{u}^* + \frac{1}{Pr} \mathbf{g}^*
\]

For low Re we throw out inertial terms. For high Re we throw out viscous terms (\( \mathcal{O}(\frac{1}{Re}) \)). This yields the inviscid (zero viscosity) Euler equations:

\[
\frac{\partial p}{\partial t} = -\nabla \cdot \mathbf{u} - g \mathbf{g}
\]

These equations are most useful for irrotational flows (e.g., \( \mathbf{\omega} = 0 \))

\[
\mathbf{u} = \nabla \times \mathbf{B}
\]

If a flow starts out irrotational, then only the viscous term can produce vorticity! Thus, if the flow is inviscid, it stays irrotational.

You can prove this by taking the curl of the \( \nabla - \mathbf{s} \) equations but it gets a little messy!

Anyway, if \( \mathbf{\omega} = 0 \) we get:

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \mathbf{0}
\]

If the flow is also steady:

\[
\nabla \cdot \left( \frac{1}{2} \mathbf{u}^2 + P - g \mathbf{g} \cdot \mathbf{x} \right) = 0
\]

\[
\nabla \left( \frac{1}{2} \mathbf{u}^2 + P - g \mathbf{g} \cdot \mathbf{x} \right) = \mathbf{u} \times \mathbf{\omega}
\]
How does this vary along a streamline? From Lagrangian perspective, time rate of change (for steady flow) along streamline is just:

\[ \frac{\partial u}{\partial t} \]

Thus:

\[ u \cdot \nabla \left( \frac{1}{2} u^2 + p - sg \cdot x \right) = 0 \]

\[ \frac{1}{2} u^2 + p + sg z = C \text{ const} \]

along a streamline!

(Note: \( g \cdot \nabla = (-\epsilon g \cdot \nabla) = g \cdot z \) if \( g \) is in \(-z\) direction!)

This is known as Bernoulli's eq'n, valid for steady, inviscid, irrotational flows.

Neglecting losses, what is the velocity of the jet, the force on the nozzle?

Conservation of Mass: \( U_A = U_2 A_2 \)

Conservation of mech. Energy (e.g., Bernoulli's eq'n):

\[ \frac{1}{2} g v^2 + p_1 = \frac{1}{2} g v_2^2 + p_2 \]

Thus:

\[ p_1 - p_2 = \frac{1}{2} g (v_2^2 - v_1^2) \]

\[ = \frac{1}{2} g U_1^2 \left( \left( \frac{A_2}{A_1} \right)^2 - 1 \right) \]

So:

\[ U_1 = \left[ \frac{2(p_1 - p_2)}{g \left( \left( \frac{A_2}{A_1} \right)^2 - 1 \right)} \right]^{1/2} \]

and:

\[ U_2 = \frac{A_1}{A_2} U_1 \]

What is the physical interp. of Bernoulli's eq'n? Conservation of Mechanical Energy? If we have no frictional losses (e.g., \( \mu = 0 \) \( \Rightarrow \) inviscid flow) then mechanical energy is conserved along a streamline!

\[ \frac{1}{2} g u^2 = \text{ kinetic Energy} \]

\[ p + sg z = \text{ "Potential Energy"} \]

Thus, if one goes up, the other goes down!

How can we use this? Look at a jet of water at high Re:

\[ A_1, U_1, P_1 \longrightarrow A_2, U_2, P_2 \]

This assumes that the flow field is uniform across inlet & outlet & that there are no frictional losses.

What about the force on the nozzle? We did this sort of problem before:

\[ \int (g u) \cdot \nabla \Delta A = \sum \text{ (force exerted on fluid)} \]

We are interested in \( x \)-component (flow direction), thus:

\[ \Sigma F_x = \int (g u) \cdot \nabla \Delta A = g U_1 (-U_4 A_1) \]

\[ + g U_2 U_1 (U_2 A_2) \]

\[ = g \left( U_2 A_2 - U_1 A_1 \right) = g U_1^2 A_1 \left( U_2 A_2 \right) \left( \frac{A_2}{A_1} - 1 \right) \]

\[ = g U_1^2 A_1 \left( \frac{A_2}{A_1} - 1 \right) \]
But from Bernoulli's eqn:
\[ \frac{\nabla u_i}{2} = \frac{2 (P_i - P_e)}{(\frac{A_1}{A_2})^2 - 1} \]

So:
\[ \sum F_x = 2 A_1 (P_i - P_e) \frac{(\frac{A_1}{A_2})^2 - 1}{(\frac{A_1}{A_2})^2 - 1} \]
\[ = 2 A_1 (P_i - P_e) \frac{A_1}{A_1 + A_2} = 2 A_1 A_2 (P_i - P_e) \frac{A_1}{A_1 + A_2} \]

Now \( \sum F_x = F_N + PA_1 - P_e A_2 \)

So:
\[ F_N = PA_1 - P_e A_2 - 2 A_1 A_2 (P_i - P_e) \frac{A_1}{A_1 + A_2} \]

Now if \( P_e = 0 \) (atmospheric pressure forces on nozzle are neglected) then:
\[ F_N = PA_1 \left(1 - \frac{2 A_2}{A_1 + A_2}\right) \]

So we have:
\[ \frac{1}{2} \rho u_e^2 = P_e - P_e \]
To solve, we need the radial velocity everywhere under the plate!
This, in turn, gives us the pressure!
By conservation of mass:
\[ 2 \pi r h u(r) = 2 \pi R h u_e \]
\[ u(r) = \frac{R}{r} u_e, \text{ at least for } r > R_0 \]
We can take \( u = 0 \) for \( r < R_0 \) (stagnation flow - it's a bit approximate!)

So:
\[ \frac{1}{2} \rho u_e^2 + P_e = \frac{1}{2} \rho u^2 + P \]

or:
\[ P = P_e + \frac{1}{2} \rho u_e^2 (1 - \frac{u^2}{u_e^2}) \]
\[ = \left(P_e + (P_o - P_e) \left(1 - \frac{R_e}{R_o}\right) \right) \text{ at } r < R_0 \]
\[ \text{or } r < R_0 \text{ (stagnation)} \]

Let's look at a more complicated problem: what are the forces on a plate near a spool of thread as depicted below:

What happens? Can we blow the plate off the spool of thread?

First, what is \( u_e \)? We shall assume inviscid, irrotational flow. Thus:
\[ \frac{1}{2} \rho u_e^2 + P_e = \frac{1}{2} \rho u_0^2 + P_o \]

To get the net force on the plate, we need to integrate:
\[ F = \int_0^{R_1} (P - P_e) 2 \pi r dR \]
\[ = (P_o - P_e) \pi R_0^2 + \int_0^{R_1} (P_o - P_e) \left(1 - \frac{R^2}{R_o^2}\right) dR \]
\[ = \pi R_0^2 (P_o - P_e) \left(1 - 2 \ln \left(\frac{R}{R_o}\right)\right) \]

So if \( 2 \ln \frac{R}{R_o} > 1 \) the net force drives the plate towards the spool.
The harder you blow, the tighter it sticks! The critical ratio is \( \frac{R}{R_o} > 1.65 \)

Bernoulli problems offer lots of interesting, counter-intuitive examples like this!
So \( \phi \) satisfies Laplace's eqn! Such problems are easy to solve for many geometries!

Problems for which \( \nabla \cdot \mathbf{u} = 0, \nabla^2 \mathbf{u} = 0 \) are known as ideal potential flow, and occur for steady, inviscid, irrotational flow fields!

Let's work a classic example - flow past a cylinder.

\[
\begin{align*}
\nabla \cdot \mathbf{u} &= 0, \quad \nabla^2 \mathbf{u} = 0 \\
\end{align*}
\]

In cylindrical coordinates:

\[
\frac{\partial \phi}{\partial r} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \phi^2} = 0
\]

How do we solve this? Look at inhomogeneous B.C.'s (those at \( r = \infty \)).

They suggest a solution of the form:

\[
\phi = f(r) \cos \theta
\]

We plug into B.C.'s:

\[
\begin{align*}
\frac{\partial \phi}{\partial r} &= \frac{f'}{r} \cos \theta - f \sin \theta \\
\frac{\partial \phi}{\partial \theta} &= -f' \cos \theta - f \sin \theta \\
\phi |_{r = \infty} &= 0 \\
\phi |_{r = 0} &= U \\
\end{align*}
\]

Both are satisfied if \( f \sim U \) as \( r \to \infty \).
Plugging into \( V^2 \phi = 0 \):
\[
\cos \theta \frac{d^2 \phi}{dx^2} + \cos \theta \frac{d \phi}{dx} - \frac{n}{2} \cos \theta = 0
\]

or
\[
\frac{d^2 \phi}{dx^2} + \frac{d \phi}{dx} - \frac{n}{2} = 0
\]

By B.C.'s:
\[
\phi \bigg|_{x=0} = 0 \quad \text{and} \quad \phi \bigg|_{x=\infty} = 0
\]

\( \phi \) is of the form:
\[
\phi = e^{nx} \frac{C_1}{x} + C_2 x
\]

From condition as \( x \to \infty \), \( C_2 = 0 \)

At \( x = a \) we have:
\[
\phi \bigg|_{x=a} = \left[ \frac{C_1}{x} + U \right]_{x=a} = 0
\]

Thus \( C_1 = U a^2 \)

And hence:
\[
\phi = U \left( \frac{a^2}{x} + r \right) \cos \theta
\]

This yields the velocity distribution:
\[
u_\theta = -\frac{a^2}{x} \frac{d \phi}{dx} = U \left( 1 - \frac{r^2}{a^2} \right) \cos \theta
\]

\[
u_r = -\phi \frac{d \phi}{dx} = -U \left( 1 - \frac{r^2}{a^2} \right) \cos \theta
\]

A couple of things to note. First, \( u_\theta \bigg|_{r=a} \neq 0 \). Thus the tangential velocity violates the no-slip condition, as expected! This leads to the development of a very thin

Boundary layer next to the surface, where both viscosity and no-slip condition must apply! A Reynolds number based on the thickness of the boundary layer is of \( \text{O}(1) \)!

\[
\frac{U a}{\nu} = \frac{U a}{\nu} > 1
\]

Seconly, \( u_\theta \bigg|_{r=a} = 2U \sin \theta \), which varies from zero at the leading and trailing stagnation points to twice the free stream velocity at

\[
\theta = \frac{\pi}{2} \quad \text{This means the fluid is accelerated going around the cylinder, and thus the pressure is lowest at } \theta = \frac{\pi}{2}! \text{ Let's calculate this:}
\]

We have Bernoulli's eqn:
\[
\frac{1}{2} g u^2 + p + \rho g \theta = \text{const}
\]

We neglect gravity! For upstream we have \( p = p_0 \), \( u = U \) on all streamlines. Thus:
\[
\frac{1}{2} g u^2 + p = p_0 + \frac{1}{2} g U^2
\]

at \( r = a \) \( u = U \) \( (u_r = 0) \), thus:
\[
p \bigg|_{r=a} = p_0 + \frac{1}{2} g U^2 (1 - \frac{r^2}{a^2} \sin^2 \theta)
\]
Thus the drag for ideal potential flow around a cylinder is zero! This is known as D’Alembert’s Paradox, and arises because the pressure distribution is symmetric—there is high pressure on both the front and back sides, which cancel out!

What really happens? => You don’t get pressure recovery on the back side!

The Boundary Layer separates, & no longer is attached to the boundary! This results in a much higher drag! => Separation is critical for high lift flows! Consider flow past a wing:

The AP from top to bottom provides lift which makes the plane fly!
If there is no separation, the drag is quite low! It’s the drag that the engines have to overcome to keep the plane moving! A commercial airliner has a max L/D ratio of ~20!

What happens if the B.L. separates? This will happen if the plane moves too slowly, or at too large an angle of attack:

Separation does two things. First, it greatly increases drag, decreasing the L/D ratio and, since engines aren’t designed to overcome this force, the plane slows down! Since \( L = \frac{1}{2} C_D U^2 \), slowing down the plane kills the lift, and the plane falls! Second, wing control surfaces (e.g. elevators) are on the trailing edge of the wing. If the...
Flow separates, these surfaces are now in a separation bubble and can no longer control the motion of the plane. This whole process is called stall and a huge part of wing design is figuring out how to avoid it.

$N-S$ equations in this region!

Let's look at a simple problem: high-Re flow past a plate of length $L$ at zero incidence (e.g., edge-on into flow):

$$
\begin{align*}
\frac{D^2 u}{Dx^2} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad &\text{near} & \quad x \ll L \\
\frac{D u}{Dx} &= 0 \quad &\text{for} & \quad x \gg L
\end{align*}
$$

If we have $Re = \frac{UL}{v} \gg 1$ (Re = plate Reynolds based on length $L$), we get the Euler flow equations:

$$
\frac{D^2 u}{Dx^2} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \frac{\partial^2 u}{\partial y^2} \quad \text{small}
$$

The BC is just $u \frac{\partial u}{\partial y} = 0$ (no normal flow)

Because we've eliminated viscosity, we've also eliminated the No-Slip condition!

Boundary Layer Theory

The scaling of the $N-S$ equations at high $Re$ suggests that viscous terms are unimportant on a length scale comparable to the size of a body. The Euler flow equations which result require eliminating the no-slip condition! This leads to discontinuities in the velocity at the surface, thus viscous forces must be important in this region, termed the boundary layer: the region where inertia and viscosity are equally significant.

We can determine the thickness of the B.L. by rescaling the

For the plate $(y=0)$ we have the undisturbed flow:

$$
\frac{\partial u}{\partial y} = U \delta_x
$$

This set of equations has the simple solution:

$$
\frac{\partial u}{\partial y} = U \delta_x \quad \text{everywhere!}
$$

But this leads to a step change in the velocity at $y=0$ (the plate). Since viscous forces are proportional to velocity derivatives, they must become important in this region!

Suppose viscous forces are important over some region $y = O(\delta)$. We shall rescale the $N-S$ equations to preserve the viscous term & the No-Slip condition.
Let: \( x = X, u = U \)
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]

To determine \( u \) & \( v \) we must look at the equations. First (always) we do the C.E.: 
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]

Thus we require:
\[
U = \frac{L}{S} U
\]

Which is the same scaling we got in lubrication theory!

Now for the X-momentum eign:
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]

Let \( U_L \), now we scale:
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]

Dividing through:
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]

We want to keep a viscous term! Thus we require:
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]

What does this mean? Basically, a boundary layer is too thin to support a pressure gradient in the y-direction. The pressure distribution due to the external Euler (inviscid) flow is imposed on the boundary layer.
This applies equally well to other boundary layer problems, such as flow past a cylinder, etc. In these flows we take $x$ to be the coordinate along the surface (e.g., $x = a\theta$ for a cylinder of radius $a$) and $y$ to be the coordinate normal to the surface (e.g., $y = r - a$ for the same geometry):

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

Ok, let's return to the flat plate problem. We have the B.C.'s:

**B.C.**:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

For the flat plate problem,

\[ u^* \rightleftarrows 1 \quad \text{(const)} \] and \[ p^*_{\text{EF}} = 0 \]

For steady state flow \[ \frac{\partial u}{\partial x} = 0 \] so:

\[ \frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} = 0 \]

\[ u^* \frac{\partial u^*}{\partial x} + v^* \frac{\partial u^*}{\partial y} = \frac{\partial u^*}{\partial y} \]

\[ u^* \left|_{y=0} \right. = 0 \quad u^* \left|_{y=\infty} \right. = 1 \]

How do we solve this set of equations? The flow is $2-D$, so it is natural to define a stream function:

\[ u^* = \frac{\partial \psi}{\partial y}, \quad v^* = \frac{\partial \psi}{\partial x} \]

Substituting in:

\[ \frac{\partial^2 \psi}{\partial y^2} \frac{\partial^2 \psi}{\partial x^2} \quad \frac{\partial^2 \psi}{\partial y^2} \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial y^2} \frac{\partial^2 \psi}{\partial x^2} \]

With B.C.'s:

\[ \frac{\partial \psi}{\partial y} \left|_{y=0} \right. = 0 \quad \frac{\partial \psi}{\partial y} \left|_{y=\infty} \right. = 1 \]

We still have a 2nd order non-linear PDE. What can we do with it??

This sort of problem often admits a similarity transform which allows us to convert a PDE to an ODE, a tremendous simplification! How do we know if this will happen? Apply Morgan's Theorem.

1) If a problem, including B.C.'s, is invariant to a one-parameter group of continuous transformations then the number of independent variables may be reduced by one.
2) The reduction is accomplished by choosing as new dependent and independent variables combinations which are invariant under the transformations.

The techniques for applying this theorem can be quite messy, but we'll stick to the simplest one: simple affine stretching.

Let's stretch all of the dependent variables! Let:
\[ y_\ast = A y, \quad x_\ast = B x, \quad y = C y \]

where \( A, B, C \) are a group of stretching parameters. If the problem can be made invariant while leaving one of those undetermined,

it will satisfy Morgan's Theorem!

Let's be this. Plugging in:
\[ \frac{A^2}{BC^2} \left( -\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x \partial y} \right) = \frac{A}{C^2} \frac{\partial \phi}{\partial y} \]

where subscripts denote derivatives.

Dividing thru:
\[ \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y^2} = \frac{B}{AC} \frac{\partial \phi}{\partial y} \]

Thus the equation is invariant if \( \frac{B}{AC} = 1 \) (e.g., \( A, B, C \) disappear!)

We also have to look at the B.C.'s:
\[ \left. \frac{\partial^2 \phi}{\partial y \partial x} \right|_{y=0} = 0 \quad \Rightarrow \quad \left. \frac{\partial \phi}{\partial y} \right|_{y=0} = 0 \]

(no restrictions)

Similarly, \( \left. \frac{\partial \phi}{\partial y} \right|_{y=0} = 0 \)

What will work? In general, any combination of variables which is invariant under the transformations will work, but some are better than others!!

For example, we have the transform:
\[ y_\ast = A y, \quad x_\ast = B x, \quad y = C y \]

and the restrictions:
\[ \frac{B}{A C} = 1 \quad \Rightarrow \quad \frac{C}{A} = 1 \]

Thus one possibility is:
\[ x_\ast = f (y) \quad \Rightarrow \quad y = \frac{y_\ast}{x_\ast} \]

which is clearly invariant! This would work, but would be extremely messy to use, with lots of implicit differentiation required! A better
choice is to respect the restrictions so that the variable \( y^* \) only involves dependent variables.

We had:
\[
\frac{B}{AC} = 1, \quad \frac{C}{A} = 1
\]

A more convenient pair of restrictions is obtained by division:
\[
\frac{A}{C} = 1, \quad \frac{B}{C^2} = 1
\]

which yields the transform:
\[
\frac{y^*}{x^*} = f(z), \quad z = \frac{x}{y^2}
\]

This works better, but it's still not the best choice! The problem is that we are taking 5th derivatives with respect to \( y^* \), but only 1st

Derivatives w.r.t. \( x^* \), if this makes sense to put all the complexity in \( x^* \):
\[
\frac{A}{Bv_0} = 1, \quad \frac{C}{Bv_0} = 1
\]

yields:
\[
\frac{y^*}{x^*} = f(z), \quad z = \frac{x^*}{v_0}
\]

(the factor of \( 2 \) and \( y^* \) are there for historical reasons - it gets rid of the constant in the transformed DE and has no significance! What matters is the dependence on \( y^* \) and \( x^* \)).

This is known as Canonical Form:
Put all the complexity in the variable with the lowest highest derivative.

There can be exceptions to this for special problems, but it usually works pretty well.

\[\begin{align*}
\frac{\partial^2 y}{\partial x^2} &= \frac{2}{(2x)^3} \left( -\frac{y}{(2x)^2} \right) - \frac{1}{2} \frac{y'}{x} \frac{\partial y}{\partial x} = -\frac{1}{2} \frac{y'}{x} \\
\frac{\partial^3 y}{\partial x^3} &= \frac{1}{(2x)^4} f, \\
\frac{\partial^2 y}{\partial x^2} &= \frac{1}{2x^2} f''
\end{align*}\]

But:
\[
\frac{\partial^2 y}{\partial x^2} = \frac{2}{(2x)^3} \left( \frac{y}{(2x)^2} \right) - \frac{1}{2} \frac{y'}{x} \frac{\partial y}{\partial x} = -\frac{1}{2} \frac{y'}{x}
\]

Thus:
\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{(2x)^4} \left( f - 2 f' \right)
\]

and finally:
\[
\frac{\partial^2 y}{\partial x^2} = \frac{2}{(2x)^4} \left( f' - 2 f'' \right)
\]

Ok, now we plug back into the DE:
\[
\frac{\partial^4 y}{\partial x^4} = \frac{2}{(2x)^5} \left( f' - 2 f'' \right) - \frac{1}{2x} \frac{\partial y}{\partial x} = \frac{2}{(2x)^5} f''
\]

which simplifies to:
\[
f'' + \frac{2}{x} f' = 0
\]

This is known as the Blasius Equation.

For flow past a flat plate!
We also have the B.C.'s:

\begin{align*}
    u^* |_{y^* = 0} &= 0, \\
    v^* |_{y^* = 0} &= f'(0), \\
    v^* |_{y^* = 1} &= -2v^* \frac{\partial v^*}{\partial x^*} |_{y^* = 0} = \frac{1}{2} \frac{\partial v^*}{\partial x^*} (f - 2f'), \\
    f(0) &= 0.
\end{align*}

Now since \( f'(0) = 0 \) we get

\[ f(0) = 0. \]

Finally,

\[ u^* |_{y^* = 1} = f'(\infty), \]

\[ f'(\infty) = 1. \]

So the complete problem reduces to the non-linear ODE:

\begin{align*}
    u^* &\frac{\partial u^*}{\partial x^*} = f'(\infty), \\
    v^* &\frac{\partial v^*}{\partial x^*} = \frac{1}{2} \frac{\partial v^*}{\partial x^*} (f - 2f'),
\end{align*}

where

\[ u^* |_{y^* = 0} = v^* |_{y^* = 0} = 0, \quad y^* |_{y^* = 1} = f'(\infty). \]

\[ f'(\infty) = 1. \]

\[ f(0) = 0. \]

We have to solve the ODE, which can be done numerically, but we know \( \beta_{50\%} \) will be some \( O(1) \) constant (the actual value is \( \beta_{50\%} = 1.096 \)).

What \( y \) value is this?

\[ \beta_{50\%} = \frac{\gamma_{50\%}}{(2.5)^2} = \frac{\gamma_{50\%}}{(2.5)^2} \]

Thus \( \gamma_{50\%} = (2.5)^2 \beta_{50\%} \).

So, within some \( O(1) \) number, we reach 50\% of the free stream velocity at \( y \sim \frac{(2.5)^2}{\beta_{50\%}} \) - and we get this without solving the equation.

What about the drag on the plate? Remember that we can break drag into two pieces:

\[ \mathbf{F} = \mathbf{F}_n + \mathbf{F}_t, \]

\[ \mathbf{F}_n \quad \text{normal forces} \quad \mathbf{F}_t \quad \text{shear forces} \]

(fore recoil) (skin friction)

In this case, normal forces are zero, thus we just get skin friction!

The skin friction is the shear stress:

\[ 2\omega = 2\gamma_{\infty} \bigg| y = 0 = \mu \left( \frac{\partial u}{\partial y} + \frac{2v^*}{y^*} \right) \]

\[ \text{wall shear stress at plate} y = 0. \]

So \( 2\omega = \mu \frac{2v^*}{\partial y} |_{y = 0} = \mu \frac{2v^*}{\partial y} |_{y = 0} \)

\[ = \frac{\gamma_{50\%}}{(2.5)^2} \mu \left( f'(0) \right) \]

where, plugging in for \( x^* \) & \( \gamma \), yields

\[ 2\omega = \frac{\partial}{\partial x^*} \left( \frac{\gamma_{50\%}}{(2.5)^2} \right) f''(0) \]

where \( f''(0) \) is again some \( O(1) \) constant.
Which must be calculated numerically. Doing this, we get $f''(0) = 0.040\%$, so:

$$v_\infty = 0.332 \mu \left( \frac{U}{8x} \right)^2$$

We may define a local drag coefficient:

$$C_D^{(c)} = \frac{2v_\infty}{\frac{1}{2} \rho U^2}$$

Kinetic/Coil of Flow

$$C_D^{(c)} = \frac{0.664}{(8x)^{1/2}} \approx 0.664 \frac{1}{Re_x^{1/2}}$$

local plate Re

so the drag decreases as we move down the plate. This makes sense because the B.C. is getting thicker, so the shear rate is going down. What is the total drag?

$$\frac{F}{\frac{1}{2} \rho U^2 L W} = \frac{1}{L} \int_0^L C_D^{(c)} \, dx$$

$$= \frac{1}{L} \int_0^L 0.664 \left( \frac{8x}{U} \right)^{1/2} \, dx = \frac{1.328}{Re_x^{1/2}}$$

or:

$$\frac{F}{\frac{1}{2} \rho U^2 L W} = \frac{\frac{1}{2} f''(0)}{Re_x^{1/2}}$$

In which we could have gotten everything to within some unknown O(1) cost without having ever solved the ODE!

This is the power of both scaling analysis and similarity transforms. The former tells you how a problem depends on the parameters involved, while the latter tells you about the functional form.

Ok, for flow past a flat plate we had a uniform Euler flow. What happens for a more complicated system?

Let's look at stagnation flow produced by a jet impinging on a surface (often used in cleaning).

First we look at the Euler flow: the flow is inviscid and irrotational, so:

$$\xi = -\xi \phi, \quad \nabla^2 \phi = 0$$

In this coordinate system we have:

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}$$

With B.C. $v|_{y=0} = 0$ (zero normal velocity)

Now for inviscid stagnation flow the solution is very simple:

$$u = \lambda x, \quad v = -\lambda y$$

which yields the potential:

$$\phi = -\frac{1}{2} \lambda (x^2 - y^2)$$

We will also need the pressure at the surface $y=0$. Let the pressure at the origin be $P_0$.

Since the flow is inviscid we have Bernoulli's eqn:

$$p + \frac{1}{2} \rho (u^2 + v^2) = \text{const} \text{ along a streamline}.$$ The surface $y=0$ is a streamline, and at $x=0$ the velocity vanishes. Thus:

$$P|_{y=0} = P_0 - \frac{1}{2} \rho \lambda^2 x^2$$

All this is for Euler flow. What about
the flow in the boundary layer?

We have the B.L. eqns:
\[ \frac{u}{\partial x} + \frac{v}{\partial y} = -\frac{1}{\lambda} \frac{\partial u}{\partial x} + \frac{v}{\partial y} \]

where \( \frac{v}{\partial y} \) term. We also have the B.C.'s:
\[ u, v \text{ at } y = 0 ; u \text{ at } y \to \infty \]

and \( p \) is given by Bernoulli's eqn.

outside the BL:
\[ p + \frac{1}{2} \rho u^2 = c^2 \]

\[ \frac{v}{\partial x} + \frac{\rho}{\partial y} \frac{u}{\partial x} + \rho = 0 \]

\[ \frac{1}{\lambda} \frac{\partial p}{\partial x} = -x \quad \text{(pressure decreases in x-direction)} \]

in the B.L.:
\[ \frac{u}{\partial x} + \frac{v}{\partial y} = \frac{x}{\lambda} + \frac{\partial u}{\partial y} \]

Let's define the stream function \( \psi \):
\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \]

Thus:
\[ \psi_x = \frac{x}{\lambda} \quad \psi_y = \frac{1}{\lambda} \]

\[ \psi_x \bigg|_{x=0} = 0 \quad \psi_y \bigg|_{y=0} = 0 \]

What about inhomogeneous B.C.?
\[ \frac{A}{C} \frac{\partial \psi}{\partial x} \bigg|_{y=0} = B \]

\[ \frac{A}{C} \frac{\partial \psi}{\partial y} \bigg|_{y=0} = B \]

or \( \frac{A}{BC} = 1 \) which is the same restriction as we already had!

So we only have 2 restrictions, and we'll get a similarity transform. What is it?
\[ \frac{A}{BC} = 1 \quad \frac{A}{BC} = 1 \quad \frac{C = 1 \quad \frac{A}{BC} = 1} \]

Thus:
\[ \psi_x = f(\xi) \quad \xi = \frac{x}{B} \]

\( \xi \) is not a function of \( x \), we could have guessed this because it wasn't a function of \( L \) either!
So \[ y_1 = x^2 f(y) \]
\[
\frac{2x}{y} = f(y) ; \quad \frac{2x}{y} = x^2 f', \text{ etc.}
\]
We get the transformed DE:
\[
(x^2 f') (f') - (f) (x^2 f'') = x^2 x f''
\]
or, rearranging,
\[
f'' = \frac{f'}{x} - \frac{f'}{x} - 1
\]
and \( f(0) = f'(0) = 0, f'(\infty) = 1 \)

The shear stress (which is what leads to cleaning the surface!) is just:
\[
\tau_u = \frac{U}{y} \frac{\partial u}{\partial y} \bigg|_{y_0} = -\frac{\mu \lambda x f''(0)}{y_0}
\]
where \( \mu \) is some constant!

It can be shown that any BL flow where \( u_* \sim x^k \) will admit a similarity solution.

Thus in the boundary layers:
\[
\frac{\partial u}{\partial x} = \frac{1}{\alpha} \frac{\partial u}{\partial \theta} (U) \bigg|_{y_0} = -\frac{\mu \lambda x f''(0)}{y_0}
\]
Thus for \( 0 < \theta < \frac{\pi}{2} \) the pressure gradient is negative. This means it is a source of momentum in the BL, and retards BL growth.

For \( \theta > \frac{\pi}{2} \) we have \( \frac{\partial u}{\partial x} > 0 \), so it is a sink of momentum in the BL. This leads to rapid growth, and ultimately to BL separation.

To drive a BL against an adverse pressure gradient (\( \frac{\partial u}{\partial x} > 0 \)) you have to get momentum in somehow. For laminar BL's this occurs only due to viscous diffusion (\( \frac{\partial u}{\partial \theta} \)), which is weak. A more efficient method is by promoting turbulence, since (as we'll see next lecture!) this leads to an eddy viscosity many times that of the molecular viscosity. This is done on airplane wings by vortex generators; tiny little fins that stick up out of the wing surface. These have the effect of increasing skin friction (which is small) but decreasing form drag by delaying or preventing separation.

Another example: baseballs! For a smooth sphere, the EF drag is zero because of complete pressure recovery on the back side! In practice, BL separation kills off the
recovery and leads to a drag which scales as:

\[ F \approx C_D \left( \frac{1}{2} \eta u^2 \frac{S}{A} \right) \]

where \( C_D \) is a function of Re.

We can plot up \( C_D \) vs. Re:

\[
\log C_D \approx -\frac{1}{2} \log \text{Re} \quad \text{for Re \geq 5000}
\]

The abrupt transition at \( \text{Re} = 3 \times 10^5 \) results from the transition of the BL to turbulence, delaying separation, giving an increase in pressure recovery and reducing drag.

By the seems, if the ball is thrown without rotation, it can cause it to dart sideways in an unpredictable manner due to recovery on one side, and not the other!

What about boundary layer flow on a more complex shape such as a wing? Again, define boundary layer coordinates:

\[
x = \text{distance along surface from leading stagnation point}
\]

\[
y = \text{distance normal to surface}
\]

If \( \delta \ll \delta L \), we may ignore curvature in the boundary layer. We thus get the B.L. equations in Cartesian

coordinates:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}
\]

where \( p \) is obtained from Bernoulli's law applied to the Euler (inviscid) flow outside the B.L. Let \( u_0, p_0 \) be the velocity & pressure far upstream

and let \( u_0 \) be the inner limit of the EF solution (e.g., the EF velocity evaluated at the surface).

Thus:

\[ p = p_0 + \frac{1}{2} \rho u^2 - \frac{1}{2} \rho u_0^2 \]

and thus:

\[ \frac{\partial p}{\partial x} = -\rho u_0 \frac{dx}{dy} \]

We also have the B.C.'s:

\[ u \bigg|_{y=0} = u_0, \quad u \bigg|_{y=\delta} = u_0 \]

and the C.E.:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

We may eliminate \( v \) by integrating over the B.L.:

\[ v = -\int_0^y \frac{\partial u}{\partial x} \, dy \quad \text{since} \ v \bigg|_{y=0} = 0 \]
Thus:
\[ w \frac{\partial w}{\partial x} - \left( \int_0^y \frac{\partial u}{\partial x} \, dy \right) \frac{\partial u}{\partial y} = u_0 \frac{\partial u_0}{\partial x} + \frac{2}{3} \frac{\partial u}{\partial y}, \]
with B.C.'s:
\[ u \bigg|_{y=0} = 0, \quad u \bigg|_{y \to \infty} = u_0. \]

Even with knowledge of \( u_0(x) \) (e.g., the EF solution) we still have to solve this numerically. For anything other than power law forms \( u_0(x) \) we won't get a similarity solution either.

We can develop a more useful expression, known as the integral BL form, by integrating over the BL thickness in the \( y \)-direction:
\[ \int_0^{y_f} u \frac{\partial u}{\partial x} - \left( \int_0^y \frac{\partial u}{\partial x} \, dy \right) \frac{\partial u}{\partial y} \, dy = \int_0^{y_f} u \frac{\partial u}{\partial y} \, dy \]
inside the integral. The whole LHS becomes:
\[ \text{LHS} = \int_0^{y_f} \left[ \frac{2}{3} \frac{\partial u}{\partial x} - u \frac{\partial u}{\partial x} - u_0 \frac{\partial u_0}{\partial x} \right] \, dy + \frac{u_0^2}{\partial x} - u_0 \frac{\partial u_0}{\partial x} \, dy. \]

We thus define two integrals:
\[ \frac{S}{\partial x} = \int_0^{y_f} (1 - \frac{u}{u_0}) \, dy = \text{displacement thickness}, \]
\[ \frac{\Theta}{\partial x} = \int_0^{y_f} \frac{u}{u_0} (1 - \frac{u}{u_0}) \, dy = \text{momentum thickness}. \]
Both have units of length. The displacement thickness is the distance streamlines outside the B.L. are reflected by the wedge of slow-moving fluid in the boundary layer.

The ratio \( \frac{S}{\Theta} \) is known as the shape factor and is a dimensionless measure of the shape of the B.L. velocity profile. For laminar flow past a flat plate:
\[ \frac{S}{\Theta} \approx 2.59. \]

But this will change for \( u_0 \) not constant, and if we have turbulent flow!

OK, what's all this good for? Let's look at the RHS:
\[ \text{RHS} = \int_0^{y_f} \frac{2}{3} \frac{\partial u}{\partial y} \, dy = \frac{M}{\partial x} \left( \frac{\partial u}{\partial y} \right) \bigg|_0^{y_f} = \frac{2}{3} \frac{\partial u}{\partial y} \bigg|_0^{y_f}. \]
But this is just the shear stress at the surface: \( \tau = \frac{\partial u}{\partial y} \mid_{y=0} \)

So:
\[
\frac{\tau}{S} = \frac{u}{S} \left( u \frac{\partial^2 \theta}{\partial x^2} \right) + \frac{\partial u}{\partial x} \frac{\partial \theta}{\partial x}
\]

which is known as the von Kármán boundary-layer momentum balance. In general, it's very difficult to measure a velocity derivative \( \frac{\partial^2 \theta}{\partial x^2} \) at a surface so instead we use integrals of \( u \) to get \( \theta \) & \( S \) and then use those to calculate skin friction.

For our flat plate problem \( \eta = 0 \) & \( \theta \) thus in this case:
\[
\frac{\tau}{S} = \frac{u}{S} \frac{\partial \theta}{\partial x}
\]

Biscuit problem

From the Navier-Stokes equations, they look like this:

1. Recirculation vortices superimposed on mean flow
2. Unstable laminar flow: 2-D waves and vortex formation
3. Bursting of vortices and growth of fixed turbulent spots
4. Fully developed turbulent boundary layer flow which, like flow along a sufficiently long flat plate, brings us to a discussion of turbulence!

Turbulence
Turbulent flow is chaotic and time dependent, so it is difficult to describe directly using the N-S equations. Instead, we look at the time-average of the motion!

Let \( \overline{u} = \overline{u} + u' \)

where \( \overline{u} = \frac{1}{T} \int_{0}^{T} u \, dt \)

\( u' \) is the fluctuation.

E.g., we average \( u \) over some small interval of time. By definition, the fluctuations average out:

\[
\frac{1}{T} \int_{0}^{T} u^{'} \, dt = 0
\]

The objective is to develop a set of averaged equations for \( \overline{u} \), \( \overline{E} \).
First, we look at the C.E.:

\[ \nabla \cdot \mathbf{u} = 0 \]

In general, the linear terms don't give us any trouble! It's the non-linear ones that cause problems.

Let's look at the N-S eqns:

\[ \frac{\partial \mathbf{u}}{\partial t} + \left( \mathbf{u} \cdot \nabla \right) \mathbf{u} = -\nabla p + \nabla \cdot \tau \]

Let's time average each term:

\[ \frac{1}{l} \int_{t}^{t+\Delta t} \frac{\partial \mathbf{u}}{\partial t} \, dt = \frac{1}{\Delta t} \left[ \mathbf{u}(t+\Delta t) - \mathbf{u}(t) \right] \]

\[ = \frac{1}{\Delta t} \left[ \mathbf{u}(t+\Delta t) - \mathbf{u}(t) \right] \]

Now, the second term may be non-zero, but it will have zero mean on average, and shouldn't contribute to the flow. If the time scale for turb. fluctuations is short with respect to the time scale for mean variations (e.g., the time scale of increasing or decreasing flow rates through a pipe) then the first term reduces to:

\[ \frac{1}{l} \left[ \mathbf{u}(t+\Delta t) - \mathbf{u}(t) \right] \approx \frac{\partial \mathbf{u}}{\partial t} \]

Next look at pressure:

\[ \frac{1}{l} \int \nabla p \, dt = \nabla \bar{p} \]

...and the viscosity term:

\[ \frac{1}{l} \int \nabla \cdot \tau \, dt = \mu \nabla \bar{\mathbf{u}} \]

So, the linear terms didn't cause any trouble. Now for the non-linear...
cross streamlines $\Rightarrow$ since they physically carry momentum, mass & energy, if they cross streamlines you get a flux of these quantities! You can use this to estimate the viscosity of a gas, for example!

In turbulence, Prandtl's idea was that eddies do the same thing! As two eddies exchange places (across streamlines) they also lead to momentum transfer (e.g., the Reynolds stress). In a channel, these arguments lead to:

$$\tau_{yx} = \frac{\nu}{L} \left( \frac{\partial u}{\partial y} \right)_x$$

The quantity above is the eddy viscosity by analogy with Newton's law of viscosity! The quantity $L$ is the length scale of the eddies, and the shear rate $\left| \frac{\partial u}{\partial y} \right|_x$ is the rate with which such exchanges take place!

Prandtl made the further approximation: eddies are bigger in the middle of a pipe than near the wall, so let:

$$L \approx y$$

where the wall is at $y = 0$. This constant $\kappa$ is known as the von Karman constant and is about $\kappa = 0.36$ by fitting to empirical data!

Ok, now let's apply this to flow near a wall. If the shear stress is constant, we get:

$$\tau_{yx} = \tau_{0x} = \tau_{0x} = \tau_{0x}$$

In general, $\frac{\tau_{0x}}{\tau_{0x}} \approx 2$ (about 40%!), so we'll ignore the linear contrib.

We find, empirically, the following picture:

![Image of turbulent flow profile]

In the turbulent core:

$$u_* \sim x^2 \frac{\tau_{0x}}{\nu} = \frac{u_*}{y} \left( \frac{\tau_{0x}}{\nu} \right)$$

So:

$$\frac{\nu}{\partial y} = \frac{1}{x} \left( \frac{\tau_{0x}}{\nu} \right)$$

Let's render this dimensionless! The scaling for velocity is the...
Thus once we reach the turbulent core only 520 μm from the wall, virtually the entire tube is turbulent! In general, for smooth tubes:

\[ \frac{V_+}{\sqrt{\frac{\mu}{\rho}}} = C_\lambda \frac{1}{\sqrt{Re}} \]

\[ \frac{\nu}{\sqrt{\frac{\mu}{\rho}}} = \frac{C_{\lambda \nu}}{\sqrt{Re}} \]

For \( 2 \times 10^4 < Re < 10^5 \), which provides a convenient way of estimating the thickness of the viscous sublayer (about 5 - 20 times this value).

Suppose we are pumping water through a 4" (10 cm) dia pipe at \( \langle u \rangle = 1 \text{ m/s} \). We have:

\[ Re = \frac{\langle u \rangle D}{\nu} \approx 10^5 \]

At this Re we are well into the turbulent regime! Empirical correlations suggest that for \( 2 \times 10^3 < Re < 10^5 \) the wall shear stress is about:

\[ \frac{\tau_w}{\frac{1}{2} \rho \langle u \rangle^2} \approx 0.079 \frac{1}{Re^{1/4}} \]

Thus \( \tau_w \approx 2.2 \text{ dyn/cm}^2 \) - about 2.7 x greater than would be the case for laminar flow! We thus get the friction velocity \( V_f = \left( \frac{\tau_w}{\rho} \right)^{1/2} \approx 4.7 \text{ cm/s} \) and the viscous length:

\[ \frac{\nu}{V_f} \approx 0.002 \text{ cm} \approx 20 \mu \text{m} \]
we get \( 7 - 3 = 4 \) dimensions cases. 

We can pick these a number of ways, but let's look for ones that make sense. We choose the aspect ratios: 

\[
\frac{1}{b} : \frac{a}{b}
\]

And the Reynolds number 

\[
\text{Re} = \frac{\rho u d}{\mu}
\]

The last one is the dimensional pressure. Usually we're at high Re, so use inertial scaling: 

\[
\frac{AP}{\frac{1}{2} \rho u^2} = f_\text{Re} \frac{a}{b} \frac{u}{b} \text{Re}
\]

Lp known as the other Lp.

It's actually more convenient to define 

\[
h = \frac{AP}{\frac{1}{2} \rho u^2}
\]

The loss is hydrostatic head due to fluid friction

\[
f_\text{Re} = \frac{b}{h}
\]

Thus: 

\[
f_\text{Re} = f_\text{Re} \left( \frac{a}{b} \frac{u}{b} \right) \text{Re}
\]

Empirically, we observe that for \( \text{Re} > 1 \) we have \( h \propto L \) (e.g., double the pipe length you double the pressure drop). Thus: 

\[
\frac{b}{h} = \frac{1}{b} \frac{u}{h} f_\text{Re} \frac{a}{b} \text{Re}
\]

We can define the Fanning Friction Factor: 

\[
f_\text{f} = \frac{b}{h} f_\text{Re}
\]

where 

\[
f_\text{Re} = f_\text{Re} \left( \frac{a}{b} \frac{u}{b} \right) \text{Re}
\]

If we determine \( f_\text{f} \) either theoretically or empirically, it's easy to get the head loss.

Let's look at low Re first. For laminar flow we get Poiseuille's law: 

\[
AP = 32 \frac{\mu u L}{d^3}
\]

Thus: 

\[
h = \frac{AP}{\frac{1}{2} \rho u^2} = 16 \frac{\mu u L}{d^3}
\]

\[
f_\text{f} = 16 \frac{1}{32} = 0.5
\]

Note that \( f_\text{f} \) is inversely proportional to \( \text{Re} \) as \( \text{Re} \to 0 \)! This is because we've used inertial scaling for AP, whereas at low Re, AP is viscous (viscosity scaling).

Empirically, for laminar flow \( f_\text{f} \) isn't a strong function of \( \text{Re} \), provided \( \text{Re} < 1 \). In fact, for \( \text{Re} \to 0 \) we can show that the correction is \( O(\text{Re}) \).

using the Minimum Dissipation Theorem. This will not be true at higher \( \text{Re} \), where even very small \( \text{Re} \) can play a big role by promoting turbulence.

Ok, how about turbulent flow? We start with the Law of the Wall obtained by Prandtl & von Kármán: 

\[
\bar{u} = 2.5 \ln \left[ y^+ \right] + 5.5 \text{ in the turbulent core}
\]

\[
\text{Re}^+ < 10 \text{ (Kármán's value)}
\]

Remember \( \text{Re}^+ = \frac{\bar{u}}{v} \) \( \text{Re}^+ \) is friction velocity.

Let's assume that this applies throughout the pipe, and use it to calculate \( \text{Re}^+ \)!

First, we need to relate \( y^+ \) to \( \text{Re} \): 

\[
y = \frac{y^+}{\text{Re}^+}
\]
So, \[ y^2 = \left(\frac{3}{8} \frac{W}{D} \right)^2 (R - r) \]

Now \[ u^2 > \frac{1}{\pi} \int_{r}^{R} v u \, dr \]

\[ = \left(\frac{3}{8} \frac{W}{D} \right)^2 \left[ 5.5 + 25.1 \left( \frac{20}{D} \right)^2 (R - r) \right] \]

\[ = \left(\frac{3}{8} \frac{W}{D} \right)^2 \left[ 5.5 \ln \left( \frac{20}{D} \right)^2 + 1.75 \right] \]

we need to relate \( c_0 \) to \( \Delta P \). We do this with a force balance on the pipe:

\[ c_0 \text{ TOTAL} = \Delta P \pi R^2 \]

forces must balance, so:

\[ c_0 \text{ TOTAL} = \Delta P \pi R^2 \]

\[ \text{Re} \] for wall are:

\[ \frac{c_0}{u} = \frac{\Delta P}{\pi R^2} \]

\[ \frac{\Delta P}{\pi R^2} \]

\[ \text{forces must balance} \]

\[ \text{viscous sublayer} \]

\[ \text{many plots of } f_c \text{ vs. } \text{Re} \text{ are available, but the most useful correlations are:} \]

\[ f_c = \frac{16}{\text{Re}} \text{ for } \text{Re} < 2100 \]

\[ f_c \approx 0.07 \left( \frac{D}{L} \right)^{0.8} \text{ for } 3 \times 10^3 < \text{Re} < 10^5 \]

\[ \frac{1}{\sqrt{f_c}} = 4.0 \log_{10} \left\{ \frac{\text{Re}}{\text{Re}} \right\} - 0.40 \text{ for } \frac{D}{L} = 0 \]

\[ \frac{1}{\sqrt{f_c}} = 4.0 \log_{10} \left\{ \frac{\text{Re}}{\text{Re}} \right\} + 2.28 \text{ for } \frac{D}{L} \geq 10 \]

In a pipe system we don’t have just pipe, but we also have fittings. These also contribute to the head loss. We may define, for high Re flow,

\[ \rho = \frac{\Delta P}{\pi R^2} \]

\[ \rho = \frac{\Delta P}{\pi R^2} \]

or \( \rho > 7 \Rightarrow \Delta P > 0 \), when the wall roughness sticks up outside the pipe.
Consider the pump system:

Suppose we have all 4” ID smooth pipe, and we want a flow rate $Q = 424 \text{ft}^3/\text{min}$.
What is the required power of the pump?
We have:

- 565’ of 4” pipe
- 3 90° elbows
  - 1 sudden contraction
  - 1 sudden expansion

First we calculate the $Re$:

$$Re = \frac{uD}{\nu} = \frac{Q}{\frac{\pi D^2}{4} \nu} = 424 \times \frac{12}{2 \times 10^{-6} \times 0.03} = 0.585$$

Re = 2,472 $10^5$ so flow is turbulent.

For the Re, $f_e = 0.0038$

Thus for the pipes:

$$h_L = \frac{(\Delta P)}{(550)} \left( \frac{\text{Re}}{1000} \right)^{0.75} = 25.4 \text{ ft}$$ which is nearly 1 atm!!

What about the fittings?

For a 90° elbow, we have $K = 0.7$

For a sudden contraction, we have:

$$K_{\text{contraction}} = 0.45 (1 - \beta)$$

where $\beta = \frac{A_{\text{small}}}{A_{\text{large}}}$

Here $\beta = 0$ so $K_{\text{contraction}} = 0.45$

For an expansion, we have:

$$K_{\text{expansion}} = (1 - \beta)^2 = 1$$

(based on $u$ in smaller pipe)

Thus:

$$h_L = 2 (0.7 + 0.45) \times \frac{180^2}{2 \times 100} = 3.6 \text{ ft}$$

Ok, so what is the total head loss?

It’s just the sum of the change in elevation, $h_e$ pipes & $(h_L)_{\text{fittings}}$:

$$Ah = h_2 - h_1 + (h_L)_{\text{pipes}} + (h_L)_{\text{fittings}}$$

$$= 150 + 25.4 + 3.6 = 179 \text{ ft}$$

(Adjusted by change in elevation)

What is the power requirement?

$$W = Q \Delta h g = 7800 \text{ ft} \text{lb}/\text{s} = 14 \text{ hp}$$

So we need a pump output of 14 hp.

The input will be greater due to pump inefficiencies! What pump to use?

We look for a pump that puts out

$$424 \text{ ft}^3/\text{min} = 20 \text{ ft}^3/\text{s}$$ with a $\Delta h$ of 177 ft = 54.6 m

The pump curve of a pump which would do the job is on next page:
Note that the operating point selected is close to the "BEP" curve. Best Efficiency Point. As you move away from this curve, the efficiency goes down. In other words, the efficiency is highest at the BEP. On the other hand, this means that the best point for operation is at the point on the curve that is closest to the BEP. This is the point where the maximum efficiency is achieved.

For our system, the application of the pumps: 

\[ h = \frac{3.84 \text{ ft} - 5 \text{ ft}}{30.1 \text{ ft}} = 8.6 \text{ m} \]

This is greater than the height of the pumps. This means that the system is not working at its operating condition, and it is possible that the pressure will reduce. This is why it is important to ensure that the system is working at the correct pressure.

Consider the pump application. Using the head loss approach, you will need to calculate the pressure and the water flow rate at the operating point. This will give you the flow rate and the pressure at the outlet. You can then compare this with the pressure at the inlet to determine if the system is working properly.

The trick is to find the pump which can provide the required head (170 ft) and the desired flow rate (2,000 GPM). The pump operating at 1,800 rpm produces 1,000 ft-lbf for power consumption of air compressor (100 hp). The power consumption of the air compressor is directly related to the efficiency of the pump. The gross efficiency is then given by: 

\[ \text{efficiency} = \frac{\text{workin}}{\text{workout}} \]

or about 50% efficiency - not too bad!
Ohm's Law gets modified due to the non-linear dependence of header loss on Q! As in circuits, the sum of the head loss along each possible fluid path from a common node to a common node must be the same. Let's apply this:

First, label the streams:

\[ h_1 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow h_e \]

TecTing, say,

\[ h_L = \left( \frac{(2g_f) \left( \frac{l}{D} \right)}{2g} + \frac{Q_1^2}{2g} \right) \left( \frac{Q_1}{A_1} \right)^2 + \frac{Q_1}{A_1} \]

\[ h_L = \left( \frac{(2g_f) \left( \frac{l}{D} \right)}{2g} + \frac{Q_2^2}{2g} \right) \left( \frac{Q_2}{A_2} \right)^2 + \frac{Q_2}{A_2} \]

\[ h_L = \left( \frac{(2g_f) \left( \frac{l}{D} \right)}{2g} + \frac{Q_3^2}{2g} \right) \left( \frac{Q_3}{A_3} \right)^2 + \frac{Q_3}{A_3} \]

**Index Notation** (4)

What is index notation? It is simply a compact and convenient way of representing scalars, vectors, and tensors. It is particularly useful for fluid mechanics, especially (as we shall see) at low Re.

> There is no new physics associated with index notation, however it can reveal symmetries & relations which were already there!

For any tensor, the order of the tensor is given by the
number of unrepeated \textit{indices}!

\begin{itemize}
\item[a.] \textit{No indices, scalar}
\item[b.] \textit{One index, both are vectors}
\item[c.] \textit{Two indices, 2nd order tensor}
\item[d.] \textit{Three indices, 3rd order tensor}
\end{itemize}

The letters used as subscripts don't matter, e.g. \( x_i, x_j, x_k \), etc. are equivalent

\( \Rightarrow \) exception: in an equation, each term must have the same \textit{unrepeated} indices, e.g.

\[ x_i = y_i \text{ is same as } x = y \]

but \( x_i = y_j \) is an \textit{error}!

\textbf{You cannot repeat an index in any product more than once:}

\[ x_i y_j z_i \equiv y(x \cdot z) \text{ (ok)} \]

\[ x_i y_j z_i \equiv \text{error}! \]

The order of multiplication (dot product) is preserved by the names/order of the indices!

\textbf{Remember} \( A^T x = b \)?

In index notation:

\[ A_{ij} x_j = b_i \]

To take the transpose, just reverse the order:

\[ (A_{ij})^T = A_{ji} \]

\textbf{A key feature of index notation} is the \textit{dot product}:

\( \Rightarrow \) \textit{Repeated indices} (in a product) implies summation!

Thus:

\[ x_i y_j \equiv x \cdot y = \sum_i x_i y_j \]

(e.g., \( x_1 y_1 + x_2 y_2 + x_3 y_3 \))

Just think of how you would code it up on a computer using loops!

The \textit{vector composition} or \textit{outer product} is also simple:

\[ A = xy \text{ is given by } A_{ij} = x_i y_j \]

Since there are two \textit{unrepeated indices}, \( x_i y_j \) is a 2\textit{nd} order tensor!

\textbf{Remember the Normal Equations?}

\[ A^T A \mathbf{x} = A^T \mathbf{b} \]

We would write this as

\[ A_{ki} A_{kj} x_j = A_{ki} b_k \]

We could also look at the residual from linear regression:

\[ r_i = A_{ij} x_j - b_i \]

\[ r_i = (A_{ij} x_j - b_i)(A_{ik} x_k - b_i) \]

Note that there are \textit{no unrepeated} indices in the product, so it's a scalar and that we switched a pair of \( j \)'s to \( k \)'s to avoid repeating \( j \) too many times! \( j \) was \textit{repeated}. 
already, so this is OK, e.g. 

\[ x_j x_j = x_j x_j \]
while \( x_j \neq x_k \), both are scalers

\[ \rightarrow \text{unrepeated!} \]

We define a couple of things:

\[ \chi \equiv \frac{\partial}{\partial x_i} \text{ (gradient operator)} \]

\[ T \equiv \delta_{ii} \text{ (kronecker \& \ identity matrix)} \]

\[ \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \]

Note: \( \frac{\partial x_i}{\partial x_j} = \delta_{ij} \) (identity matrix)

\[ \frac{\partial x_i}{\partial x_j} = \delta_{ij} = 1 + 1 = 2 \]

Note that we combined the two middle terms since

\[ A_{jk} x_k \equiv A_{ij} x_i \]

The use of \( k \) or \( l \) was indeterminate because they were \( \text{repeated} \), only the \( \text{unrepeated index "i"} \) has to be the same on both sides!

Ok, now we take some derivatives. Note that \( A_{ij} \) and \( b_j \) are \( \text{constants} \), so they pop out!

\[ \nabla (x_j x_j) = A_{jk} A_{ik} \frac{\partial}{\partial x_j} (x_k x_k) \]

\[ -2 A_{jk} b_j \frac{\partial x_i}{\partial x_j} + \frac{\partial b_j}{\partial x_j} \]

\[ \delta_{ki} \]

OK, let's use this to solve for the normal equations!

Recall we had \( \nabla (x_j x_j) = 0 \)

In index notation:

\[ \frac{\partial}{\partial x_i} \{ (A_{jk} x_k - b_j)(A_{jl} x_l - b_j) \} \]

\[ = \frac{\partial}{\partial x_i} \left\{ A_{jk} x_k A_{jl} x_l - b_j A_{jk} x_k - A_{jk} x_k b_j + b_j b_j \right\} \]

Or, since we only have to preserve the order of the indices:

\[ = \frac{\partial}{\partial x_i} \left\{ A_{jk} A_{ij} x_k x_l - 2 A_{jk} b_j x_k + b_j b_j \right\} \]

Now we compute the first term:

\[ \frac{\partial}{\partial x_i} (x_j x_k) = x_k \frac{\partial x_j}{\partial x_i} + x_j \frac{\partial x_k}{\partial x_i} \]

\[ = x_k \delta_{ik} + x_j \delta_{ik} \]

So:

\[ \nabla (x_j x_j) = A_{jk} A_{ik} (x_k \delta_{ik} + x_j \delta_{ik}) \]

\[ -2 A_{jk} b_j \delta_{ik} \]

Taking the dot product of a matrix (or vector) with the identity matrix leaves it unchanged. In index notation this is:

\[ A_{ij} \delta_{jk} = A_{ik} \]

(just replace the "i" with a "k")
So:
\[ \nabla (r_T^2) = A_{jk} A_{ij} x_k + A_{ik} A_{jk} x_l \]

Now the first two terms are identical since in both cases "k" and "l" are repeated indices and thus indeterminate.
So:
\[ \nabla (r_T^2) = 0 \]

becomes:
\[ 2 A_{ij} A_{jk} x_k - 2 A_{ij} b_j = 0 \]

or
\[ A_{ij} A_{jk} x_k = A_{ij} b_j \]

Which is the same as:
\[ A^T A x = A^T b \]

In addition to the \( \delta_f^a \), there is another special case we'll use:

Note that just as
\[ A \times B = -B \times A \]

In index notation we have:
\[ E_{ijk} = -E_{ijk} \]

Switching order throws in a (-1)!

If \( E_{ijk} \) is cyclic, \( E_{ijk} \) must be counter-cyclic (and vice versa).

Technically, any matrix for which \( A_{ij} = A_{ji} \) is termed symmetric.

A matrix for which \( B_{ij} = -B_{ji} \) is anti-symmetric.

Note: The double dot product (e.g., \( A_{ij} B_{ij} \) — no repeated indices) of a symmetric & an anti-symmetric
Another useful concept is isotropy. Mathematically, a tensor is isotropic if it is invariant under rotation of the coordinate system. Physically, it’s isotropic if it looks the same from all directions.

A sphere is isotropic, a football isn’t!

All scalars are isotropic.

No vectors are isotropic!

The most general 2nd order isotropic tensor is \( \lambda \delta_{ij} \)

\( \Rightarrow \) const. scale

The most general 3rd order tensor is \( \lambda \delta_{ijk} \).

Thus:

\[
\mathbf{\nabla} \times (\mathbf{\nabla} \times \mathbf{U}) = \nabla (\nabla \cdot \mathbf{U}) - \nabla^{2} \mathbf{U}
\]

What’s \( \varepsilon_{ijk} \) ?

4 unrepeated indices, so it’s a 4th order tensor.

\( \varepsilon_{ijk} \) is isotropic, so the product is also isotropic.

Hence:

\[
\varepsilon_{ijk} \nabla \times \mathbf{U} = \lambda_1 \delta_{ij} \delta_{km} + \lambda_2 \delta_{ik} \delta_{jm} + \lambda_3 \delta_{im} \delta_{jk}
\]

where \( \lambda_1, \lambda_2, \lambda_3 \) are to be determined.

We can calculate these by multiplying both sides by each of the three terms on the RHS (one at a time!) which then yields three eqns for the three \( \lambda \)’s.

The most general 4th order isotropic tensor is:

\[
A_{ijkl} = \lambda_1 \delta_{ij} \delta_{kl} + \lambda_2 \delta_{ik} \delta_{jl} + \lambda_3 \delta_{il} \delta_{jk}
\]

where \( \lambda_1, \lambda_2, \lambda_3 \) are scalars.

We can use this to prove vector calculus identities.

From texts, we have:

\[
\mathbf{\nabla} \times (\mathbf{\nabla} \times \mathbf{U}) = \nabla (\nabla \cdot \mathbf{U}) - \nabla^{2} \mathbf{U}
\]

Let’s prove this!

\[
\mathbf{\nabla} \times (\mathbf{\nabla} \times \mathbf{U}) = \varepsilon_{ijk} \nabla_{i} \left( \frac{\partial U_{j}}{\partial x_{k}} \right)
\]

Note the order of the indices.

This is important when working with \( \varepsilon_{ijk} \).

So:

\[
\varepsilon_{ijk} \nabla \times \mathbf{U} = \lambda_1 \delta_{ij} \delta_{km} + \lambda_2 \delta_{ik} \delta_{jm} + \lambda_3 \delta_{im} \delta_{jk}
\]

This is zero because \( \varepsilon_{ijk} \) is zero for all \( i, j, k \). Also, \( \varepsilon_{ijk} \delta_{ij} = 0 \) because \( \varepsilon_{ijk} \) is anti-symmetric and \( \delta_{ij} \) is symmetric, so the double-dot product of the two is likewise zero!

Now for the RHS:

\[
(\lambda_1 \delta_{ij} \delta_{km} + \lambda_2 \delta_{ik} \delta_{jm} + \lambda_3 \delta_{im} \delta_{jk}) \times (\delta_{ij} \delta_{km}) = \lambda_1 \delta_{ij} \delta_{sk} + \lambda_2 \delta_{im} \delta_{jk} + \lambda_3 \delta_{im} \delta_{sk}
\]

\[
= \lambda_1 (2) (3) + \lambda_2 (3) + \lambda_3 (3)
\]

Since \( \delta_{ij} = 1 + 1 = 3 \).!
We thus get the first equation:
\[ 0 = 9\lambda_1 + 3\lambda_2 + 3\lambda_3 \]
Now for the second term. We multiply both sides by \( \delta_{ij} \).

We get:
\[ E_{ijk} E_{kjm} \delta_{ij} = 3\lambda_1 + 9\lambda_2 + 3\lambda_3 \]
where the RHS was calculated the same way as before.

The LHS is:
\[ E_{ijk} E_{kij} \]
Now if \( E_{ijk} \) is cyclic, so is \( E_{kij} \). Likewise, if \( E_{ijk} \) is counter-cyclic, so is \( E_{kij} \). Thus, the product is just \((1)(1) = 1\) or \((-1)(-1) = 1\) for all six non-zero elements.

\[ = 3\delta_{ijm} \frac{\partial u_m}{\partial x_j} - \delta_{im} \frac{\partial u_j}{\partial x_m} \frac{\partial u_m}{\partial x_i} \]
\[ = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \]
\[ = \nabla (\mathbf{u} \cdot \mathbf{n}) - \nabla^2 \mathbf{u} \]

which completes the identity!

The last concept we wish to explore is the difference between pseudo-tensors and physical tensors. This distinction arises from the choice of right-handed or left-handed coordinate systems. A pseudo-tensor is one whose \( \delta_{ij} \) depends on this choice, a physical tensor is one which doesn’t!

This yields:
\[ \sigma = 3\lambda_1 + 9\lambda_2 + 3\lambda_3 \]
Like wise, the multiplication by the last term yields:
\[ E_{ijk} E_{jkm} \delta_{ij} = 3\lambda_1 + 3\lambda_2 + 9\lambda_3 \]
\[ = E_{ijk} E_{kji} = -6 \]
These equations have the solution \( \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -1 \)

Thus:
\[ E_{ijk} E_{kjm} = \delta_{ijm} \delta_{jm} - \delta_{im} \delta_{jm} \frac{\partial u_m}{\partial x_j} \frac{\partial u_m}{\partial x_i} \]
and hence:
\[ \nabla \times (\mathbf{u} \times \mathbf{n}) = E_{ijk} E_{jkm} \frac{\partial u_m}{\partial x_j} \frac{\partial u_m}{\partial x_i} \]

Let’s look at some examples:
- **physical** velocity, force, stress
- **pseudo** angular velocity, torque, vorticity

\( \mathbf{u} \): \( \mathbf{u} \times \mathbf{n} = \mathbf{u} \times \mathbf{u} \)

we go from one to the other via the cross-product!

The vorticity is defined as:
\[ \mathbf{\omega}_l = E_{ijk} \frac{\partial u_k}{\partial x_j} \] (e.g. \( \omega = \mathbf{\omega} \times \mathbf{n} \))
\( \mathbf{\omega}_l \) is a pseudo-vector
\( \mathbf{u}_k \) is a physical vector

Likewise,
\[ 2 \mathbf{\omega}_l = E_{ijk} \frac{\partial u_k}{\partial x_j} \] is a physical vector.
In fact, our vector...
Identity yields
\[ \nabla \times \mathbf{E} = \epsilon_{ijk} \frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial x_k} \right) \mathbf{E} = \epsilon_{ijk} \frac{\partial^2}{\partial x_j \partial x_k} \mathbf{E} \]
\[ = \epsilon_{ijk} \epsilon_{klm} \frac{\partial^2}{\partial x_l \partial x_m} \mathbf{E} \]
which is a physical vector.
The reason why we make this distinction is that a physical tensor and a pseudotensor can never be equal!

How can we use this? Consider the following problem. Suppose we have a body of revolution whose orientation is specified by the unit vector \( \mathbf{p} \), e.g.

There is only one way to do this!!
\[ \mathbf{A}_{ik} = \lambda \epsilon_{ijk} \mathbf{p}_j \]
where \( \lambda \) is some scalar!

Thus \( \mathbf{J}_i = \lambda \epsilon_{ijk} \mathbf{p}_j \mathbf{F}_k \) and a single experiment can determine \( \lambda \), which is constant for all orientations!

Likewise, if the object has fore-and-aft symmetry (e.g., a football, which looks the same for \( \mathbf{p} \) and \(-\mathbf{p}\) orientations) we have that \( \mathbf{A}_{ik} \) must be an even function of \( \mathbf{p} \). Since the only possible form of \( \mathbf{A}_{ik} \) is odd in \( \mathbf{p} \), \( \lambda \) must be zero for such objects!

Thus, in example, rods (fore-and-aft symmetric cylinders) don’t rotate when settling at low \( \mathbf{Re} \), regardless of orientation.

We can also look at the settling velocity \( \mathbf{U}_i \) (physical vector) for some \( \mathbf{F} \):
\[ \mathbf{U}_i = B_{ij} \mathbf{F}_j \]
here \( B_{ij} \) is a physical tensor which depends on \( \mathbf{p} \). The most general form is:
\[ B_{ij} = \lambda_1 S_{ij} + \lambda_2 \mathbf{p}_i \mathbf{p}_j \]
Thus:

\[ U_i = (\lambda_1 S_{ij} + \lambda_2 P_i P_j) F_j \]

where \( \lambda_1 \) & \( \lambda_2 \) must be determined from experiment or (nasty) calculation. Actually, by measuring the settling velocity of a rod broadside on and end on, you can get \( \lambda_1 \) & \( \lambda_2 \); allowing you to calculate \( U \) for all orientations - including the lateral velocity for inclined rods! We'll do this experiment later this semester.