CHEG 544 Transport Phenomena I
Second Hour Exam

Closed Books and Notes

Problem 1). (20 points) An infinite cylinder of radius R is filled with fluid. Initially the fluid is at rest. At time t = 0 the cylinder is rotated about its axis with an angular velocity which is linearly increasing in time, e.g., $u_\theta|_{r=R} = A\,t$. In this problem the velocity is only in the theta direction ($u_r = u_z = 0$) and all derivatives with respect to theta and z are zero. We're interested in the self-similar boundary layer solution here, not the solution at all times you can get another way!

a. Write down the differential equation and boundary conditions for u_θ.

b. We are interested in short times, for which the velocity will be non-zero only near the outer wall. Defining r = R - y, rewrite the equations in terms of this new variable.

c. Using δ as the length scale of the boundary layer, simplify the equations in the limit $\delta/R \ll 1$.

d. Show that the simplified equations yield a self-similar solution in this limit, obtaining the similarity rule and variable in canonical form.

e. How does the wall shear stress vary with time? How long will the simlarity solution be valid?

The Navier-Stokes equation for momentum in the theta direction is given by:

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r\,u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+\,\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r\,u_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2}\right]$$

Problem 2). (20 points) Index Notation / Creeping Flow. The far-field disturbance veolcity produced by an object undergoing net torque (but no net force) under creeping flow conditions is often called a rotlet, analogous to that produced by one undergoing a net force (but no torque) called a Stokeslet. These singularities are important: it has been proposed, for example, that rotlets (produced by cilia) are what lead to the development of left-right asymmetry in vertebrates.

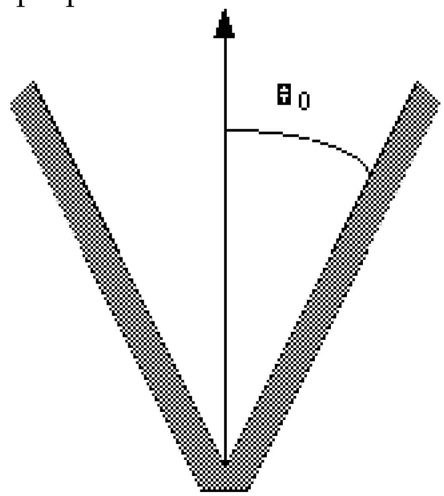
a. If the point torque is specified by the pseudovector M_i (or M_j, or M_k ...) determine this far field velocity distribution to within an arbitrary constant.

b. Recalling that torque (like force) is conserved, that the torque on any patch of surface can be calculated from $\varepsilon_{ijk}\ x_j\,\sigma_{kl}\,n_l\,dA$ (e.g., locally r X F), and the definition of the stress tensor, figure out what the constant is.

You may find the following decaying spherical harmonics useful:

$$\frac{1}{r}\ ;\ \frac{x_i}{r^3}\ ;\ \left(\frac{x_i x_j}{r^5}-\frac{\delta_{ij}}{3r^3}\right)\ ;\ \left(\frac{x_i x_j x_k}{r^7}-\frac{x_i\,\delta_{jk}+x_j\,\delta_{ik}+x_k\,\delta_{ij}}{5r^5}\right)\ ;\ \ldots$$

Problem 3). (20 points) Consider the two-dimensional geometry depicted below. A very viscous fluid is contained in a triangular trough (angle $2\theta_0$), where the walls are not moving. An elastic sheet is centered in the middle of the trough ($\theta = 0$). We want to calculate the streamfunction for the fluid flow resulting from the stretching of the elastic sheet. Note that this stretching yields the boundary condition $u_r\big|_{\theta=0} = \lambda r$, e.g., the radial velocity of the sheet is proportional to r.



a. Set up the problem as completely as possible, with all boundary conditions clearly determined.

b. Using the boundary conditions as a guide, determine the form of the streamfunction, and develop the corresponding ODE and boundary conditions.

c. Determine the general solution to the ODE, and get a set of equations for the unknown coefficients.

d. **Explicitly obtain the coefficients for the specific case where $\theta_0 = \pi/2$.**

The velocities are given by:

$$u_\theta = \frac{\partial\psi}{\partial r} \quad ; \quad u_r = -\frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

Recall that the general expression for a separable streamfunction in the cylindrical geometry is given by:

$$\psi_\lambda = r^{\lambda} f_\lambda(\theta)$$

where, in general, we have:

$$f_\lambda(\theta) = A_\lambda \sin(\lambda\theta) + B_\lambda \cos(\lambda\theta) + C_\lambda \sin((\lambda\text{-}2)\theta) + D_\lambda \cos((\lambda\text{-}2)\theta)$$

We also have the repeated root special cases:

$$f_0(\theta) = A_0 + B_0\, \theta + C_0 \sin(2\theta) + D_0 \cos(2\theta)$$

$$f_1(\theta) = A_1 \sin(\theta) + B_1 \cos(\theta) + C_1\, \theta \sin(\theta) + D_1\, \theta \cos(\theta)$$

$$f_2(\theta) = A_2 + B_2\, \theta + C_2 \sin(2\theta) + D_2 \cos(2\theta)$$

Hint: Think about how the radial velocity and streamfunction have to depend on r!