CBE 60544 Transport Phenomena I
Final Exam

Closed Books and Notes

Problem 1). (10 points) Index Notation: In the Midwest Mechanics lecture a number of years ago Keith Moffatt gave a fascinating lecture on magneto-electro-hydrodynamic migration of particles. He showed that if particles were sufficiently small that inertial forces were negligible, then the migration velocity of non-conducting particles was *Bilinear* in the current density **J** (a physical vector) and the magnetic field **B** (a pseudo vector). Bilinearity means that it is linear in the product of these two vectors. Using this observation, prove that the migration velocity of a non-conducting sphere is always orthogonal to both the magnetic field and the current, and that the rotational velocity of such a sphere is identically zero.

Problem 2). (10 points) Creeping Flow: A sphere of radius a is rotating with angular velocity Ω_i in an infinite fluid at rest.

a. Derive the fluid pressure and velocity distributions at zero Re.

b. It is proposed to examine the effect of fluid inertia (e.g., small but finite Re) as a perturbation expansion. Determine if this problem admits a regular perturbation expansion, and if not at what order do we need to resort to a singular perturbation approach? Note that I'm not asking you to actually get the perturbation velocities, just to figure out what approach we need to use!

Problem 3). (10 points) Spinning disk electrodes: Several times this semester we have talked about the "spinning disk electrode", which is simply a rapidly rotating disk immersed in a fluid at rest. For high rotation rates, the flow only occurs in a narrow layer near the rotating disk, with the pressure and velocity (theta and radial, anyway) being zero outside of this layer. The key balances are convection of theta momentum balancing diffusion of theta momentum away from the disk, and convection and diffusion of r-momentum balancing centrifugal force. You also must satisfy continuity, of course.

With this in mind, scale the momentum equations to determine the characteristic boundary layer thickness near a disk of radius R rotating with angular velocity Ω in a fluid of viscosity μ and density ρ. Note that the *mass transfer* boundary layer thickness is just going to be $(D/\nu)^{0.5}$ times this value, which makes it really thin! The Navier-Stokes equations in cylindrical coordinates are given below.

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho\, r\, u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho\, u_\theta) + \frac{\partial}{\partial z}(\rho\, u_z) = 0$$

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\,u_r)\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right] + \rho g_r$$

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r\, u_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\,u_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right] + \rho g_\theta$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right] + \rho g_z$$

Problem 4). (20 points) You are evaluating design proposals for a heating element for a microfluidic channel as depicted below. The fluid channel is of width W and depth H, and the heating section is of length L, with the usual conditions L>>W>>H. The lower wall is held at a constant temperature T_1, and the fluid enters with a temperature T_0 at flow rate Q. The upper wall is insulated (no heat flux). Such a channel might be incorporated into a chip-based PCR replication system, for example.

a. What is the asymptotic temperature distribution far down the channel (very easy!)?

b. Derive the equation governing the temperature distribution in the channel, render it dimensionless, and develop the corresponding Sturm-Liouville eigenvalue problem for the decaying part.

c. It is desired to bring the fluid up to the required temperature with as small a residence time ($t_R = V/Q$ where V is the volume of the heating element LWH) as possible. Using scaling analysis, show how the t_R required to achieve equilibrium varies with the design parameters of the problem. Pressure drop is also a concern. For a given flow rate Q, how does the pressure drop Δp vary with the required residence time t_R and channel width W (Note: you can use the relation between t_R and H to eliminate dependence of Δp on H)?

d. While the equation you got in part b can be easily solved numerically, you are trapped on a desert isle without access to your computer - thus you have to get an approximate answer using pencil and paper. Making any approximations you feel are necessary, give me a quantitative back-of-the-envelope answer to the question of how long the channel has to be for the flow-average temperature to have reached 95% of its asymptotic value (if you didn't bring a calculator, $e^3 \sim 20$).