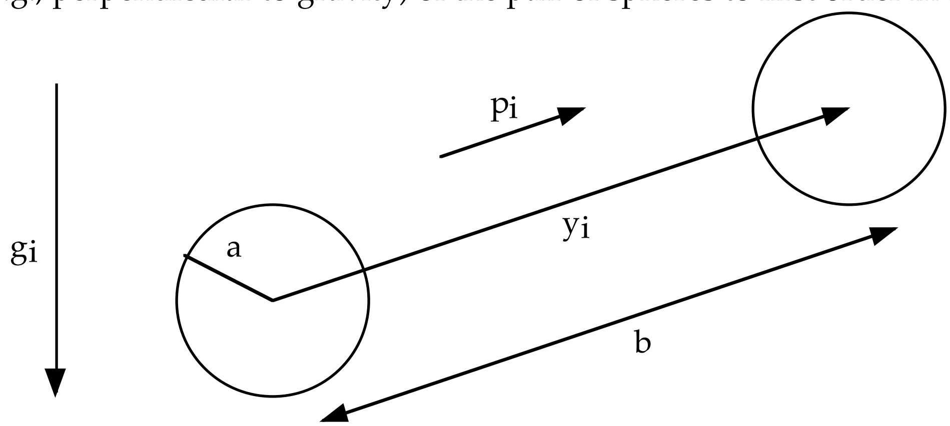
CHEG 544 Transport Phenomena I
Second Hour Exam

Closed Books and Notes

Problem 1). (10 points) Creeping Flow. A pair of identical spheres of radius a and density ρ_s are settling under the influence of gravity in an infinite fluid of viscosity μ and density ρ_f as depicted below. At time t=0 they are separated by a vector $y_i = b\, p_i$ which is not in general aligned with the gravity vector g_i. In this problem we explore their dynamics under creeping flow conditions. The motion of each sphere is, to leading order, just its Stokes flow sedimentation velocity plus the disturbance velocity produced by the other sphere. Using this, determine the maximum sideways drift velocity (e.g., perpendicular to gravity) of the pair of spheres to first order in a/b.



Hint: Recall that the velocity field produced by a Stokeslet (point force singularity of strength F_k, the far-field of that produced by a sphere) is:

$$u_i = \frac{F_k}{8\pi\mu}\left\{\frac{\delta_{ik}}{r} + \frac{x_i x_k}{r^3}\right\}$$

Problem 2). (10 points) Index Notation / Creeping Flow. Prove that the **far-field** disturbance velocity induced by any force-free and torque free particle (e.g., characterized by the stresslet $\mathbf{S_{jk}}$ which is a **physical**, **symmetric**, and **traceless** second order tensor) is purely radial. The single undetermined constant you should have the problem reduce to **could** be determined by examining the far-field velocity of some particular stresslet which you know - such as the disturbance velocity of a sphere immersed in a pure straining motion (don't do this, however!). You may find the following decaying spherical harmonics useful:

$$\frac{1}{r}\ ;\ \frac{x_i}{r^3}\ ;\ \left(\frac{x_i x_j}{r^5} - \frac{\delta_{ij}}{3r^3}\right)\ ;\ \left(\frac{x_i x_j x_k}{r^7} - \frac{x_i\delta_{jk} + x_j\delta_{ik} + x_k\delta_{ij}}{5r^5}\right)\ ;\ \ldots$$

Hint: Which of the relevant harmonics will contribute to the velocity and pressure for a stresslet in the **far-field limit**?

Problem 3). (20 points) Consider the two-dimensional problem depicted below. We are examining the flow pattern in produced by a belt in the vicinity of a roller submerged in a viscous fluid. The belt moves with a velocity U, so the radial velocity boundary condition at $\theta = -\alpha$ is -U and that at $\theta = +\alpha$ is +U. **Determine** the velocity profile using a streamfunction formulation. **Are there any conditions** under which Moffatt Eddies can occur in this geometry for these boundary conditions? **Why or why not?**

The velocities are given by:

$$u_\theta = -\frac{\partial \psi}{\partial r} \qquad ; \qquad u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

Recall that the general expression for a separable streamfunction in the cylindrical geometry is given by:

$$\psi_\lambda = r^{\lambda} f_\lambda(\theta)$$

where, in general, we have:

$$f_\lambda(\theta) = A_\lambda \sin(\lambda\theta) + B_\lambda \cos(\lambda\theta) + C_\lambda \sin((\lambda-2)\theta) + D_\lambda \cos((\lambda-2)\theta)$$

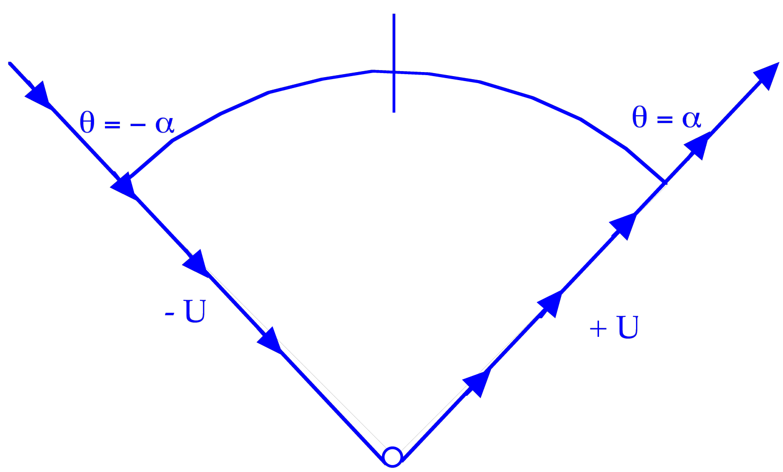
We also have the repeated root special cases:

$$f_0(\theta) = A_0 + B_0\,\theta + C_0 \sin(2\theta) + D_0 \cos(2\theta)$$

$$f_1(\theta) = A_1 \sin(\theta) + B_1 \cos(\theta) + C_1\,\theta \sin(\theta) + D_1\,\theta \cos(\theta)$$

$$f_2(\theta) = A_2 + B_2\,\theta + C_2 \sin(2\theta) + D_2 \cos(2\theta)$$

Hint: Think about how the radial velocity and streamfunction have to depend on r, and determine the symmetry conditions in θ!



Problem 4). (20 points) Lubrication: The motion of a sphere normal to a plane solved in class is a useful way of probing the surface morphology of particles, in particular measuring the effective hydrodynamic surface roughness. This only works, however, if the equations of motion apply: if the fluid in the gap between the particle and the plane does not cavitate. Cavitation, in general, will occur when the absolute pressure in the center of the gap falls to zero (this assumes the vapor pressure of the liquid is negligible). Here we examine the cavitation limitation:

a. Using scaling analysis, for a particle of radius a and density ρ_S in a fluid of viscosity μ and density ρ_S at absolute pressure P_{atm}, determine how the minimum surface roughness $h_0 = \varepsilon_S a$ measurable by this technique depends on the parameters of the problem.

b. Solve for the "order one numerical constant" in part a by solving the relevant lubrication problem.

c. A friend argues that the answers in a and b should really be modified to include the effects of surface tension Γ. Surface tension will suppress cavitation because the pressure inside the cavitation bubble will be $2\Gamma/h$ greater than the pressure outside it: thus, you won't get cavitation in a gap of width h until the pressure falls $2\Gamma/h$ below zero absolute. How does this modify the calculated minimum measurable surface roughness?

You may find the r momentum equation useful to get the pressure gradient:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r}$$
$$+ \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\, v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho\, g_r$$

Remember that in lubrication flow (at least for this problem) only a couple of terms survive!