1). Self-Similarity revealed by stretching: Consider the 2-D steady diffusion of a solute away from a point source in a shear flow. The solute is released from the wall at the origin \((x, y = 0)\) at a rate \(Q/W\) where \(W\) is the unit extension into the paper. It diffuses in the \(y\)-direction, and is convected in the \(x\) direction by the shear flow \(u = \dot{\gamma} y\). The mass flux through the wall is zero. The final condition is that the total solute convected through any plane downstream of the source must equal to the release rate. Thus:

\[
\dot{\gamma} y \frac{\partial \phi}{\partial x} = D \frac{\partial^2 \phi}{\partial y^2} \quad ; \quad \frac{\partial \phi}{\partial y} \bigg|_{y=0} = 0 \quad ; \quad \phi \bigg|_{y=-\infty} = 0 \quad ; \quad \int_0^\infty \dot{\gamma} y \phi \, dy = \frac{Q}{W}
\]

a. Render the governing equations dimensionless using some arbitrary length scale \(L\) in the \(x\)-direction. What is the characteristic concentration and boundary layer length scale in terms of \(L\) and the parameters of the problem?

b. Using affine stretching (or the results of part a), show that the problem admits a self-similar solution. Obtain the transformed ODE and boundary conditions in canonical form, and determine the concentration at the wall to within an \(O(1)\) unknown constant.

c. Solve the ODE numerically using the shooting method to determine this constant.

2). Self-Similarity revealed by trial functions: Consider the flow of a dilute emulsion (suspension of drops) flowing through a tube. My former student Arun Ramachandran (now a professor in Toronto) demonstrated that, just like the case of a dilute suspension in Couette flow, the concentration profile in a tube is also self-similar. The dimensionless governing equations for the drop concentration \(\phi\) (under a simplifying assumption) is given by:

\[
\frac{\partial \phi}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \phi \right) + \frac{\Lambda}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi^2 r}{\partial r} \right)
\]

The dimensionless parameter \(\Lambda\) is the ratio of diffusion to the inward droplet drift, and is the only dimensionless group which appears in the problem. The boundary conditions are:

\[
\phi \bigg|_{r=r_e} = 0 \quad ; \quad \int_0^{r_e} \phi \, 2r \, dr = 1
\]

and a no-flux condition at the center:

\[
\frac{\partial \left( \phi^2 r \right)}{\partial r} \bigg|_{r=0} = 0
\]
Note that the concentration using this model is predicted to be singular at the center; in real-life this singularity is relaxed by considering finite drop size effects.

a. Calculate the concentration distribution at steady-state, and determine the critical value of $\Lambda$ for which $r_c<1$ (e.g., the limit of validity of the model).

b. Show that the concentration distribution is self-similar, and obtain the complete time-dependent solution resulting from a step change in the parameter $\Lambda$ (e.g., changing the flow rate).

Hint: It is really quite similar to the Couette problem described in King & Leighton. The problem is written up in Ramachandran, Loewenberg & Leighton, POF 2010.

3). Scaling of the transport equations: Consider a cold window of height $H$ and temperature $T_0$ as depicted below. The temperature far away is a reference temperature $T_\infty$ and the medium is air. The contraction of air near the window due to the temperature difference gives rise to a buoyancy driven convection process (a cold draft down the window) and results in a steady-state heat transfer.

a. Write down the equations and boundary conditions governing the problem. Note that the equations are the same as were used for natural convection from a line source of energy given at https://www.nd.edu/~dtl/cheg459/pivexperiment although the geometry and boundary conditions are different (and simpler for your problem).

b. By scaling the equations, estimate how the total rate of heat loss from the window depends on the parameters of the problem.

c. If the window is at 0°C and the room (the air far away) is at 20°C, and the window measures 2m high and 1m wide, what is the numerical value of this estimated heat loss? If the energy cost is 0.10 $/kWhr (the current cost of electricity in Indiana), about how much does the loss from this one window cost per day?
4). In the problem above, you get the values to within an $O(1)$ constant (the unknown dimensionless derivative at the surface of the window). We can get this constant fairly easily by using a similarity transform and a little numerical integration. Here we do this!

a. Introduce the streamfunction $\psi$ such that:

$$u = \frac{\partial \psi}{\partial y} ; \quad v = -\frac{\partial \psi}{\partial x}$$

so that you “use up” the continuity equation and reduce the number of dependent variables from 3 to 2.

b. Show that the problem admits a self-similar solution via simple affine stretching. Determine the similarity rules and similarity variable in canonical form, and obtain the transformed ODEs.

c. Solve the problem numerically for the Prandtl number of air by converting it into a system of first order ODEs and using a two parameter shooting method, and compare your numerical answer with the $O(1)$ constant assumed in problem 3. Note that unlike the natural convection due to a line source of energy problem, this set of equations behaves fairly well even for large values of $\eta$. Thus, if you just integrate out to, say, $\eta = 20$ you get nice convergence for any reasonable initial guess of the unknown shooting parameters.