Let's look at another problem in cylindrical coordinates:

Coullet Flow!

we again use the \( r, \theta, z \) coord.

System. This time, however, the velocity is in the \( \theta \) direction!

\[
\nabla \cdot \mathbf{u} = 0 = \frac{1}{\rho} \frac{\partial}{\partial r} (r \ u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}
\]

Thus if \( u_r = u_z = 0 \) then \( \frac{\partial u_\theta}{\partial \theta} = 0 \)

(no variation in \( \theta \) direction)

Now for the momentum equations:

\[ \Rightarrow \text{we looked at } z\text{-momentum last time} \]

now look at \( \nu \) & \( \theta \) components!
\( r \) - momentum:

\[
S \left( \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_\phi \frac{\partial u_r}{\partial \phi} \right)
= - \frac{\partial p}{\partial r} + S g_r + \mu \left[ \frac{1}{r^2} \left( \frac{\partial}{\partial r} \left( r u_r \right) \right) \right.
+ \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial \phi^2} \right]
\]

Now if \( g_r = 0 \), \( u_r = u_\phi = 0 \) and \( \frac{\partial u_\theta}{\partial \theta} = 0 \)
we're left with:

\[
- \frac{\partial p}{\partial r} + \uparrow
\]

centrifugal force term! It is a "pseudo force" which arises from the coordinate transformation!

Thus \( P = f(\theta, z) + \int g \frac{u_\theta^2}{r} \, d\theta \)
which can be integrated if you know \( u_\theta(r) \)!
Ok, let's look at the \( \theta \) component (where the action is!)

\[
\frac{s}{\theta} \left( \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{\sqrt{-\gamma}} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_\theta^2 \frac{\partial u_\theta}{\partial \theta} \right)
\]

\[
= -\frac{1}{\nu} \frac{\partial p}{\partial \theta} + 8\nu \theta + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{\nu} \frac{\partial u_\theta}{\partial r} \right) \right]
\]

\[
+ \frac{1}{\nu^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + 2 \frac{1}{\nu^2} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_\theta}{\partial z^2} \]

Ok, most of these terms are zero too! Let's look at one that pops up due to the coordinate transformation:

\[
\frac{s}{\theta} \frac{u_r u_\theta}{r}
\]

This is the coriolis force. It is very important in large scale (e.g., high \( \text{Re} \)) rotating systems. The most important example is the weather! It's why the wind direction
is perpendicular to pressure gradients!

To see why this occurs, consider a disk undergoing solid body rotation:

\[ \theta \]

Now \( u_0 = \Omega r \) for solid body rotation. The local angular velocity is constant. If fluid is displaced *inwards*, then if \( u_0 \) is conserved (say, conservation of kinetic energy) the local rate of rotation \( \Omega \frac{u_0}{r - dr} > \Omega \). In the rotating reference frame, it looks like it's going faster!
On the earth, rotational velocities are much higher than wind velocities, at least on large length scales, thus the Coriolis force is dominant.

\[ \Omega Z R \approx \frac{2\pi}{24 \text{ hr}} \cdot 4,000 \text{ mi} \approx 10^3 \text{ mph!} \]

On large length scales it's small (at least due to earth rotation) \Rightarrow the bathtub vortex is due to some initial swirling motion!

OK, how about Couette flow? \( u_r = 0 \) so Coriolis force doesn't matter!

\[ \frac{\partial P}{\partial \theta} = 0 \quad \text{from symmetry, so:} \]

\[ 0 = \mu \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial}{\partial R} (R u_R) \right) \]
we integrate this once:
\[
\frac{1}{r} \frac{d}{dr} (ru_\theta) = C_1
\]
And a second time:
\[
u u_\theta = \frac{1}{2} C_1 r^2 + C_2
\]
or:
\[
u u_\theta = \frac{1}{2} C_1 r + \frac{C_2}{r}
\]
We have the no-slip B.C.'s:
\[
u u_\theta = \begin{cases} 
0 & r = R_0 \\
\nu 2R_1 & r = R_1
\end{cases}
\]
Thus:
\[
\frac{1}{2} C_1 R_0 + \frac{C_2}{R_0} = 0
\]
\[
\frac{1}{2} C_1 R_1 + \frac{C_2}{R_1} = \nu 2R_1
\]
So:
\[
C_1 = -2 \frac{C_2}{R_0^2} ; \ qquad C_2 = -\nu \left( \frac{R_1^2 R_0^2}{R_1^2 - R_0^2} \right)
\]
and:
\[
u u_\theta = \nu 2R_1 \left( \frac{R_1 R_0}{R_1^2 - R_0^2} \right) \left( \frac{r^2 - R_0^2}{R_0^2} \right)
\]
We wish to calculate the torque on the inner cylinder. We have:

\[ M = \tau \times F \]

Now the force \( F \) is just the shear stress \( \tau \rho \) times the area of the cylinder. Recall \( \tau \rho = F/A \) exerted by fluid of greater \( \nu \) on fluid of lesser \( \nu \) in the \( \theta \) direction!

So:

\[ M = \rho \cdot \frac{2\pi \nu h_1 \tau \rho \hat{e}_z}{\text{Area}} \]

In cylindrical coordinates:

\[ \tau \rho = \mu \left[ \frac{\partial}{\partial r} \left( \frac{u_r}{r} \right) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right] \]
Now $u_r = 0$, and $u_\theta$ is given by:

$$u_\theta = \frac{\sqrt{2} R_1^2}{R_1^2 - R_0^2} \left( 1 - \frac{R_0^2}{r^2} \right)$$

So:

$$\tau_{\theta \theta} = 2\mu \frac{\sqrt{2} R_1^2}{R_1^2 - R_0^2} \frac{R_0^2}{r^2}$$

and hence the torque:

$$M = 4\pi \mu h R \frac{R_1^2 R_0^2}{R_1^2 - R_0^2} \frac{A}{r^2}$$

Note that this is independent of $\omega$! This makes sense: the torque exerted by the outer cylinder is the same as that exerted on the inner cylinder, and every cylindrical surface in between. Otherwise the flow would be accelerating (not at steady-state)!
Ok, what about the thin-gap approximation? Just as the earth looks flat when viewed on a human length scale, so fluid mechanics problems may be simplified when characteristic lengths (e.g. the gap width between cylinders) is much smaller than the radius of curvature!

We take \( \frac{R_1 - R_0}{R_1} \ll 1 \)

Locally, we define coordinates:

\[
\begin{align*}
X &= \Theta R_0 \\
Y &= r - R_0 \\
Z &= R_1
\end{align*}
\]

The force \( F \) is approximately:

\[ F \approx 2y_x \cdot 2\pi R_0 h \]
where: 
\[ \Sigma_{y} \approx \mu \frac{\sqrt{2} R_1}{R_1 - R_0} \]

So:
\[ (M)_{\text{approx}} = \mu \frac{\sqrt{2} R_1}{R_1 - R_0} \cdot R_0 \cdot 2\pi R_0 \cdot h \cdot \hat{z} \]

We can compare this to the exact result:
\[ \frac{(M)_{\text{approx}}}{(M)_{\text{exact}}} = \frac{1}{2} \cdot \frac{R_1^2 - R_0^2}{R_1 (R_1 - R_0)} = 1 - \frac{1}{2} \frac{R_1 - R_0}{R_1} \]

So if \( R_0 = 1'' \) and \( R_1 - R_0 = 0.02'' \) (about 500\( \mu \)m), then the error is only around 1%.

---

In this derivation we have assumed that \( u_r = u_z = 0 \). This will be valid provided the rotation rate is sufficiently small. At higher
rotation rates the flow becomes unstable, yielding what are called Taylor-Couette vortices.

To see why, remember the centrifugal force term in the \( r \)-momentum equation:

\[
\frac{\partial \rho \mathbf{v}}{\partial t} = -\nabla \mathbf{p} + \mathbf{f}_{\text{centrifugal}}
\]

Because \( u_0 \) is higher inside (smaller \( r \)) than outside (larger \( r \)) the fluid inside "wants" to flow out while that outside "wants" to flow in. This produces the vortex pattern:

\[
\begin{align*}
R_0 &\quad \rightarrow \quad R_1 \\
\text{Vortices in } \mathbb{W}-Z \text{ plane}
\end{align*}
\]
Parallel Plate Flow

B.C.'s \[ u_r \mid z=0 = u_r \mid z=H = 0 \]
\[ u_z \mid z=0 = u_z \mid z=H = 0 \]

\[ u_r \mid z=H = u_z \mid z=H \]

But \[ u_\theta \mid z=H = \omega \frac{r}{2} \] (solid body rotation)

If \( u_r, u_z = 0 \) everywhere
then from CE:
\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r u_r \right) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_\phi}{\partial \phi} = 0 \]
\[ \frac{\partial}{\partial \phi} = 0 \]

axisymmetric anyway!

Look at \( \theta \)-mom eqn:
\[
\begin{align*}
&\left( \frac{\partial v_\theta}{\partial t} + u_\theta \frac{\partial v_\theta}{\partial r} + u_r \frac{\partial v_\theta}{\partial \theta} \right) \left( \frac{\partial v_\theta}{\partial \theta} \right) \\
&= -\frac{1}{r} \frac{\partial p}{\partial \theta} + g \theta + \mu \left[ \frac{2}{r^2} \left( \frac{\partial}{\partial r} (r u_\theta) \right) \right. \right. \\
&\left. \left. + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2} \right] \\
\text{If } u_r = u_\theta = 0 \text{ then } \frac{\partial^2 u_\theta}{\partial z^2} = 0.
\end{align*}
\]

Note that \( u_r = u_\theta = 0 \) so \( r \)-deriv term vanishes. Very convenient!

\[ u_\theta = A \frac{z}{r} + B \quad \text{maybe } f^\theta (r) \]

By inspection, \( B = 0, \ A = \frac{\sqrt{2}}{4} \)

so \[ u_\theta = \sqrt{2} \frac{z}{4} \]

In general, we are interested in the torque - used in viscosity measurements!
Torque = \int_0^R 2\pi r \cdot r \cdot 2\pi r^2 \, dr \\
= \int_0^R \frac{\partial u_\theta}{\partial z} 2\pi r^2 \, dr \\
= \int_0^R \frac{\partial}{\partial z} \frac{u_\theta}{2} 2\pi r^2 \, dr \\
= \frac{\pi}{2} \frac{u_\theta}{H} R^4 \\

So measure the torque, calculate the viscosity.

Ok, what about finite inertia?

Is \( u_r = 0 \)?

Let's look at radial momentum again:

\[
3 \left( \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_\theta \frac{\partial u_r}{\partial \theta} - \frac{u_r^2}{r} + u_\theta \frac{\partial u_r}{\partial \theta} \right) = -\frac{\partial p}{\partial r} + 9\pi r + M \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (ru_r) \right) \right] \\
+ \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2} 
\]
The problem is the Coriolis force!

If \( u_r, u_\theta = 0 \) then

\[-\frac{\partial u_\phi}{\partial r} = -\frac{2\Omega}{r}\]

but \( u_\phi = \frac{2r^2 \Omega}{h} \)

\[\frac{\partial P}{\partial r} = 3 r^2 \Omega \left( \frac{\Omega}{h} \right)^2 = P'(2)\]

This would produce some flow in the \( z \)-direction - it can't work!

What happens is that you get an inertial secondary current in the radial direction, and from CE a flow in \( z \)-direction as well.

We can solve for this via regular perturbation.
Parallel Disk Reg. Perturbation

Let \( u^* = \frac{u}{H} \), \( x^* = \frac{x}{H} \), \( p^* = \frac{p}{\sqrt{\mu}} \)

\[
\begin{align*}
\frac{\sqrt{\lambda H^2}}{\partial^2} \left\{ u^* \cdot x^* \cdot u^* \right\} &= -\nabla^2 p^* + \nabla^2 u^* \\
\partial^* \cdot u^* &= 0
\end{align*}
\]

Drop all \( x^* \)s, \( \lambda = \frac{\sqrt{\lambda H^2}}{\partial^2} \)

\( \varepsilon \cdot u \cdot \nabla u = -\nabla p + \nabla^2 u \)

\( \partial^* \cdot u = 0 \)

\( u_i \bigg|_{z=1} = \varepsilon \delta_{ij} \delta_{j3} x_k \)

\( u_i \bigg|_{z=0} = 0 \)

\( \varepsilon \left\{ V_\mu \frac{\partial V_\mu}{\partial r} - \frac{V_\mu^2}{\mu} + V_2 \frac{\partial V_2}{\partial z} \right\} = -\frac{\partial p}{\partial r} + \frac{\partial}{\partial r} \left( \frac{1}{\mu} \frac{\partial}{\partial r} \left( \mu V_\theta \right) \right) + \frac{\partial^2 V_\theta}{\partial z^2} \)

\( \varepsilon \left\{ V_\mu \frac{\partial V_\theta}{\partial r} + \frac{V_\mu V_\theta}{\mu} + V_2 \frac{\partial V_\theta}{\partial z} \right\} = \frac{\partial}{\partial r} \left( \frac{1}{\mu} \frac{\partial}{\partial r} \left( \mu V_\theta \right) \right) + \frac{\partial^2 V_\theta}{\partial z^2} \)

\( \varepsilon \left\{ V_\mu \frac{\partial V_2}{\partial r} + V_2 \frac{\partial V_2}{\partial z} \right\} = -\frac{\partial p}{\partial z} + \frac{1}{\mu} \frac{\partial}{\partial r} \left( \mu \frac{\partial V_\mu}{\partial z} \right) + \frac{\partial^2 V_2}{\partial z^2} \)

\[ \frac{1}{\mu} \frac{\partial}{\partial r} \left( \mu V_\mu \right) + \frac{\partial V_\theta}{\partial z} = 0 \]

\( V_\mu \bigg|_{z=0,1} = 0 \)

\( V_2 \bigg|_{z=0,1} = 0 \)

\( V_\theta \bigg|_{z=0} = \varepsilon \nu \)

\( V_\theta \bigg|_{z=1} = 0 \)
Regular perturbation in $\epsilon$:

Let $V_\theta = V_\theta^0 + \epsilon V_\theta^1 + \cdots$

$V_\nu = V_\nu^0 + \epsilon V_\nu^1 + \cdots$

$V_z = V_z^0 + \epsilon V_z^1 + \cdots$

$p = p^0 + \epsilon p^1 + \cdots$

We have the $O(\epsilon^0)$ problem:

$$-\frac{\partial p^0}{\partial \nu} + \frac{\partial}{\partial \nu} \left( \frac{1}{\nu} \frac{\partial}{\partial \nu} (\nu V_\nu^0) \right) + \frac{\partial^2 V_\nu^0}{\partial z^2} = 0$$

$$\frac{\partial}{\partial \nu} \left( \frac{1}{\nu} \frac{\partial}{\partial \nu} (\nu V_\theta^0) \right) + \frac{\partial^2 V_\theta^0}{\partial z^2} = 0$$

$$-\frac{\partial p^0}{\partial z} + \frac{1}{\nu} \frac{\partial}{\partial \nu} \left( \nu \frac{\partial V_z^0}{\partial \nu} \right) + \frac{\partial^2 V_z^0}{\partial z^2} = 0$$

with B.C.'s:

$$V_\nu^0 \bigg|_{z=0} = V_z^0 \bigg|_{z=1} = 0$$

$$V_\theta^0 \bigg|_{z=0} = 0$$

$$V_\theta^0 \bigg|_{z=1} = 1$$

This has the simple solution:

$$V_\theta^0 = \mu z, \quad V_\nu^0 = V_z^0 = p^0 = 0.$$
\[ v_\theta = \sqrt{2} \nu \frac{z^2}{4} (1 + O(\epsilon)) \]

OK, now let's examine the \( O(\epsilon^3) \) problem.

We note that \( V_r, V_z = O(\epsilon) \) only, thus:

\[ O(\epsilon^3): \]

\[ \frac{-\nu (v_\theta')^2}{\nu} = -\frac{\partial P'}{\partial r} + \frac{2}{\nu} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \nu v_r' \right) \right) + \frac{\partial^2 v_r'}{\partial z^2} \]

\[ -\nu z^2 \]

\[ 0 = \frac{\partial}{\partial r} \left( \frac{1}{\nu} \frac{\partial}{\partial r} (r \nu v_\theta') \right) + \frac{\partial^2 v_\theta'}{\partial z^2} \]

\[ 0 = -\frac{\partial P'}{\partial z} + \frac{1}{\nu} \frac{\partial}{\partial r} \left( r \frac{\partial v_z'}{\partial r} \right) + \frac{\partial^2 v_z'}{\partial z^2} \]

\[ \frac{1}{\nu} \frac{\partial}{\partial r} (r v_r' z') + \frac{\partial v_z'}{\partial z} = 0 \]

\[ v_r' \bigg|_{0}^{r_0} = v_z' \bigg|_{0}^{r_0} = v_\theta' \bigg|_{0}^{r_0} = 0 \]

Note that \( v_\theta \) has the trivial solution \( v_\theta' = 0 \).

There will be a correction to \( v_\theta \), but it will be \( O(\epsilon^2) \)!

How do we solve this? Introduce the streamfunction:
\[
\frac{1}{\nu} \frac{2\gamma}{\delta \gamma}, \quad V_\nu = \frac{1}{\nu} \frac{\partial \gamma}{\partial \nu}
\]

\[
-\nu \frac{\partial^2}{\partial z^2} = -\frac{\partial P'}{\partial \nu} + \left( \frac{1}{\nu} \left( \frac{\partial^2 \gamma}{\partial z^2} \right) \right) + \frac{1}{\nu} \frac{\partial \gamma}{\partial z} \frac{\partial \gamma}{\partial z}
\]

\[
0 = -\frac{\partial P'}{\partial \nu} + \frac{1}{\nu} \left( \frac{\partial^2 \gamma}{\partial z^2} \right) \frac{\partial \gamma}{\partial z} - \frac{1}{\nu} \frac{\partial \gamma}{\partial z} \frac{\partial \gamma}{\partial z}
\]

Take deriv. of 1st equation w.r.t. \( z \) & 2nd w.r.t. \( \nu \) and subtract to eliminate \( P' \):

\[
\left( \frac{1}{\nu} \frac{\partial^2 \gamma}{\partial z^2} \right) \frac{\partial \gamma}{\partial z} + \frac{1}{\nu} \frac{\partial \gamma}{\partial z} \frac{\partial \gamma}{\partial z} + \left( \frac{1}{\nu} \frac{\partial \gamma}{\partial z} \frac{\partial \gamma}{\partial z} \right)
\]

We anticipate a solution of the form \( V_\nu \omega^2 \)

\[
\text{try sol'}: \quad \gamma = \omega^2 f(z)
\]

\[
\frac{\partial^2}{\partial \nu^2} \left( \frac{1}{\nu} \frac{\partial^2 \gamma}{\partial z^2} \right) = \frac{\partial^2}{\partial z^2} \left( f \frac{\partial}{\partial \nu} \left( \frac{1}{\nu} \frac{2\omega^2}{\delta \omega} \right) \right) = 0
\]

and \( \frac{\partial \gamma}{\partial \nu} \left( \frac{1}{\nu} \frac{\partial \gamma}{\partial \nu} \left( \frac{1}{\nu} \frac{2\omega^2}{\delta \omega} \right) \right) = 0 \)

\[
-\omega^2 \frac{\partial^2}{\partial z^2} = \nu f''(z)
\]

\[
f''(z) = -2z
\]
We need 4 B.C.'s:

\[ V_x'(0,1) = 0 \implies f(0,1) = 0 \]
\[ V_y'(0,1) = 0 \implies f'(0,1) = 0 \]

\[ f = -\frac{2}{5 \cdot 4 \cdot 3 \cdot 2} z^5 + A z^3 + B z^2 + C z + D \]
\[ f(0) = 0 \implies A + B + C + D = 0 \]
\[ f'(0) = 0 \implies C = 0 \]
\[ f(1) = 0 \implies f(1) = -\frac{1}{60} + A + B = 0 \]
\[ f'(1) = 0 \implies f'(1) = -\frac{1}{12} + 3A + 2B = 0 \]

\[ A + B = \frac{1}{60} \]
\[ 3A + 2B = \frac{1}{12} \]
\[ A = \frac{1}{12} - \frac{1}{30} = \frac{1}{20} \]
\[ B = -\frac{1}{30} \]

So \[ f = -\frac{1}{60} z^2 (z^3 - 3\frac{1}{20} z + \frac{1}{20} z^2) \]
And:

\[ V_w = \frac{1}{r} \frac{\partial^2 y}{\partial z^2} = r f' = -\frac{1}{60} r z^2 (5 z^3 - 9 z + 4) \]

\[ V_z = -\frac{1}{r} \frac{\partial y}{\partial r} = -2f = \frac{1}{30} z^2 (z^3 - 3z + 2) \]

or, in dimensional form,

\[ V_w = \left( \frac{\sqrt{g H^2}}{w} \right) \left( \frac{r}{H} \right)^2 \left( \frac{z}{H} \right) \left( \frac{z}{H} \right)^3 \left( \frac{z}{H} \right)^2 - \frac{1}{60} \left( z^3 \right) - 9 \left( \frac{z}{H} \right) + 4 \]

and \[ V_z = \left( \frac{\sqrt{g H^2}}{w} \right) \left( \frac{1}{30} \right) \left( \frac{z}{H} \right)^2 \left( \frac{z}{H} \right)^3 - 3 \left( \frac{z}{H} \right) + 2 \]

Note that max radial velocity is at a position

\[ f''(z_{\text{max}}) = 0 \quad \Rightarrow \quad 20 z_{\text{max}}^3 - 18 z_{\text{max}} + 4 = 0 \]

\[ (z_{\text{max}})^{\text{lower}} = 0.2370 \quad \text{Not symmetric!} \]

\[ (z_{\text{max}})^{\text{upper}} = 0.8077 \]

Note that \( V_w \) max \( \approx \left( \frac{\sqrt{g H^2}}{2} \right) \left( \frac{\sqrt{w}}{H} \right) \left( \frac{0.5126}{60} \right) \)

Important in some aspects involving particle migration in PP flow.