Flow thru a tube w/ an Axial Wire
Ref: Perturbation Methods, Van Dyke

We have eq'n:
$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = \frac{1}{\mu} \frac{\partial p}{\partial z} = \text{cst}$$

render dimensionless. Let \( u^* = \frac{u}{a} \),

$$u^* = \frac{u}{\left( \frac{a^2 \partial p}{\mu \partial z} \right)}$$

$$\frac{1}{u^*} \frac{\partial}{\partial u^*} \left( u^* \frac{\partial u^*}{\partial u^*} \right) = -1$$

w/ \( u^* = 0 \) at \( r^* = 1 \), \( u^* = \tilde{z} \) where \( \tilde{z} \ll 1 \)

we may solve this directly, but let's use method of matched asymptotic exp. instead:

wire thru elliptical tube not solvable analytically, but may be solved asymptotically.

So: Problem has 2 regions => global region where \( u^* = \mathcal{O}(1) \), inner region where \( u^* = \mathcal{O}(\tilde{z}) \) & problem must be rescaled.
Look at outer region:

$\text{soln for } \varepsilon = 0 \text{ is just (dropping } \varepsilon^3\text{)}$

$$u = \frac{1}{4}(1 - r^2) + O(1) \Rightarrow \text{little } O \text{ means of smaller mag. than which is not valid as } r \to \varepsilon$$

In inner region:

let $\xi = \frac{r}{\varepsilon}$

\[ \frac{1}{\xi} \frac{\partial}{\partial \xi} \left( \xi \frac{\partial u}{\partial \xi} \right) = -\varepsilon^2 \]

To leading order in inner region, just have:

\[ \frac{1}{\xi} \frac{\partial}{\partial \xi} \left( \xi \frac{\partial u}{\partial \xi} \right) = 0 \]

\[ u = c_1 + c_2 \ln \xi \]

w/ B.C. that $u \bigg|_{\xi = 1} = 0 \Rightarrow c_1 = 0$

and $u = c_2 \ln \xi$  (note: $c_2$ may be $f''(0)$)

We must match this soln w/ outer soln as $\xi \to \frac{1}{\varepsilon}, r \to \varepsilon$

Writing inner soln in outer variables:

$$u = c_2 \ln \left( \frac{r}{\varepsilon} \right) = c_2 \left[ \ln(r) + \ln \left( \frac{r}{\varepsilon} \right) \right]$$
\[ U = C_2 \ln \left( \frac{1}{\varepsilon} \right) \left[ 1 + \frac{\ln(1 + \varepsilon)}{\ln(1/\varepsilon)} \right] \]

Now as \( \varepsilon \to 0(\varepsilon) \) outer soln \( n \) becomes

\[ U \approx \frac{1}{4} \left( U = \frac{1}{4} \left( 1 - (1 - \varepsilon^2) \right) \right) \to \frac{1}{4} + O(\varepsilon^2) \text{ in inner region} \]

\( \varepsilon \to 0(\varepsilon) \)

Thus inner soln \( n \) is of form:

\[ U = \frac{1}{4} \left( \ln \left( \frac{1}{\varepsilon} \right) \right) = \frac{1}{4} \left( 1 + \frac{\ln(1 + \varepsilon)}{\ln(1/\varepsilon)} \right) \]

We may construct a uniformly valid approx to soln \( n \) from inner \& outer soln:

\[ = \text{inner soln } + \text{outer soln} - \lim_{\varepsilon \to 0} \text{ of outer soln in inner region (to correct order)} \]

\[ U \approx \frac{1}{4} \left( 1 + \frac{\ln(1 + \varepsilon)}{\ln(1/\varepsilon)} \right) + \frac{1}{4} \left( 1 - \varepsilon^2 \right) - \frac{1}{4} \]

\[ = \frac{1}{4} \left( 1 - \varepsilon^2 \right) + \frac{1}{4} \left( \ln \left( \frac{1}{\varepsilon} \right) \right) \frac{\ln(1 + \varepsilon)}{\ln(1/\varepsilon)} \]

which is very similar to exact soln:

\[ U = \frac{1}{4} \left( 1 - r^2 \right) + \frac{1}{4} \left( \frac{\ln(1 + \varepsilon)}{\ln(1/\varepsilon)} \right) \left( 1 - \varepsilon^2 \right) \]
Matching principle

$m$-term inner exp. of $(n$-term outer exp)

$= n$-term outer exp. of $(m$-term inner exp)$

$m/n$ any integers, but usually $m=n$ or $n+1$
Now look at another problem: Unsteady cond. from a cylinder.

We wish to let \( T \) profile & \( Q \) from inf. cylinder at temp \( T_1 \) in solid by \( x \) & temp \( T_0 \) at long times.

Look at steady problem first:

\[
\frac{\partial T}{\partial t} = \alpha \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = 0
\]

let \( t^* = \frac{t \kappa}{\alpha}, \quad r^* = \frac{r}{a}, \quad T^* = \frac{T - T_0}{T_1 - T_0} \); \( t^* \to 0 \) as \( r^* \to \infty \)

Sol'n to steady problem:

\[
T^* = C_1 + C_2 \ln r^* \quad \text{at} \quad r^* = 1, \quad T^* = 1
\]

\[
T^* = 1 + C_2 \ln r^*
\]

Can't match condition at \( \infty \).

Now for unsteady problem:

convenient to introduce new time variable \( S \):

\[
S = \frac{1}{2 \beta \sqrt{t}}
\]

\( \therefore \) interested in problem when \( s \to 0 \)

Now have eq'n: (drop all *'s):

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + 2S^3 \frac{\partial T}{\partial S} = 0
\]

since \( \frac{\partial}{\partial t} = \frac{\partial}{\partial S} \frac{\partial S}{\partial t} = -2S^3 \frac{\partial}{\partial S} \)
For small \( s \) we have inner problem:

\[
\frac{1}{\nu} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = 0
\]

with inner solution: \( T = 1 + C_2 \ln \nu \)

where \( C_2 \) will be \( f^a(s) \) to be determined.

In outer region, terms

\[
\left( \frac{1}{\nu} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) \right) \quad \text{and} \quad 2s^3 \frac{\partial T}{\partial s} \quad \text{balance}
\]

\[
\frac{2}{\nu^2} > \frac{2}{s^2} T
\]

\[\therefore \text{in outer region, rescale } r \text{ w/ } s\]

and we let \( g = rs \) \( \text{render dim by } g = \frac{r}{(4\pi t)^{1/2}} \)

w/ eq'n in terms of \( g, s \): not orthogonal coordinates, which complicates matters.

\[
\left( \frac{\partial T}{\partial r} \right)_s = s \left( \frac{\partial T}{\partial s} \right)_s
\]

Similarly

\[
\left( \frac{1}{\nu} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) \right)_s = s^2 \left( \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial T}{\partial s} \right) \right)_s
\]
and we also have:

$$\left( \frac{\partial T}{\partial s} \right)_r = \left( \frac{\partial T}{\partial s} \right)_g + \left( \frac{\partial T}{\partial s} \right)_s \frac{s}{s}$$

Thus:

$$s^2 \left( \frac{1}{s^2} \left( \frac{\partial T}{\partial s} \right)_g \right) + 2s \frac{\partial T}{\partial s} + 2s^3 \frac{\partial T}{\partial s} = 0$$

or

$$\frac{T}{s^3} + \left( \frac{1}{s} + 2s \right) \frac{T}{s} + 2s \frac{\partial T}{\partial s} = 0$$

which, to leading order as $s \to 0$, gives:

$$\frac{T}{s^3} + \left( \frac{1}{s} + 2s \right) \frac{T}{s} = 0$$

with outer soln, B.C. $T \to 0$ as $s \to 0$

Integrating:

$$T_g = C_1 e^{-\left( \ln s + s^2 \right)}$$

$$T_g = C_1 e^{-s^2}$$

$$\therefore T = \int_{-\infty}^{\infty} -\frac{C_1}{x} e^{-x^2} \, dx$$

let $z = x^2$, $\frac{dx}{x} = \frac{1}{2} \frac{dz}{z}$
and 

\[ T = \int_{0}^{\infty} \frac{e^{-z}}{z} \, dz = C \, E_{1}(s^{2}) \]

we now need to match inner & outer solutions:

\[ \lim_{z \to 0} E_{1}(z) = - \log(z) + O(1) \]

\[ T = \int_{r^{2}}^{s^{2}} \frac{e^{-z}}{z} \, dz \]

\[ = -2C \log(s) + C \cdot O(1) \]

\[ = -2C \log(s) \left( 1 + \frac{\log r}{\log s} + O\left(\frac{1}{\log s}\right) \right) \]

which must match inner sol'n:

\[ T = 1 + C_{2} \log r \]

\[ \therefore C = \frac{-1}{2 \log(s)} = \frac{-1}{2 \log(Y_{s})} \]

and \[ C_{2} = \frac{-1}{\log(Y_{s})} \]

\[ \therefore \text{we have the inner solution} \]

\[ T = 1 - \frac{\log(r)}{\log(Y_{s})} \]

and the outer solution:

\[ T = \frac{1}{2 \log(Y_{s})} E_{1}((E_{1})^{2}) \]
The energy loss from the cylinder is thus:

\[ Nu = -\left. \frac{\partial T}{\partial r} \right|_{r=1} = \frac{1}{\log(ab)} = \frac{1}{\frac{1}{2} \log\left(\frac{ab}{b^2}\right)} \quad \text{as } t \to \infty \]
Solution to hole v notch problem

radius of hole = a

radius of notch = \varepsilon a

Temp. far away from hole = T_0 + AX = T_0 + Ar \cos \theta

Thus we require a solution which satisfies the condition \( T = \text{cst} \) on the hole (and notch) & which matches the sol'n at \( \infty \).

To solve, first look at outer problem, i.e. we neglect notch.

Thus \( \nabla^2 T = 0 \), \( T = T_1 \) on \( \nu = a \), \( T \to T_0 + Ar \cos \theta \) as \( r \to \infty \)

we have

\[ \nabla^2 T = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \left[ - \mu \frac{\partial^2 T}{\partial \mu^2} + (1 - \mu^2) \frac{\partial^2 T}{\partial \mu^2} \right] \]

where \( \mu = \cos \theta \)
Let's try a separation of variables solution

\[ T = T_0 + f(r) g(\mu) \]

\[ \frac{d}{dr} (r f') = -\frac{1}{q} \left[ (1-\mu^2) g'' - \mu g' \right] = \lambda \]

Solutions to \( f(r) \) are just of form \( r^n \):

\[ r^n (r f') = \lambda f \]

\[ r^n - \lambda = 0 \quad : \quad \lambda = n^2 \]

and \( (1-\mu^2) g'' - \mu g' + n^2 g = 0 \)

which is similar (but not identical) to \( \exp n \) for Legendre polynomials.

For \( n = 0 \), \( g(\mu) = 1 \) \( \{ \text{sol'n which is finite} \}

For \( n = 1 \), \( g(\mu) = \mu \) \( \text{at } \mu = 1 \)

Higher order solutions will not be necessary.

\[ \therefore T = T_0 + (c_1 r + \frac{c_2}{r}) \mu \]
Matching at \( \infty \): \( C_1 = A \)

\[
T = T_0 + \left( A r + \frac{C_2}{r} \right) \mu
\]

at \( r = a \), \( T = T_1 \) (unknown)

\[
\therefore T_1 = T_0 \quad \text{and} \quad C_2 = -a^2 A
\]

Thus we have the outer solution:

\[
T = T_0 + \left( A r - \frac{a^2 A}{r} \right) \mu
\]

with B.C. \( T = T_0 \) on the hole!

let \( T^* = \frac{T - T_0}{Aa} \), \( r^* = \frac{r}{a} \)

Thus \( T^* = (r^* - \frac{1}{r^*}) \mu \)

but this solution does not satisfy B.C. \( T^* = 0 \) on the notch. We will solve the problem in this region using cylindrical coordinates centered on the notch:

\[
\phi = \frac{3}{2} a
\]

\( \therefore \) notch given by \( s^* = 1 \)
In the inner region we have
\[ \nabla^2 T^* = 0 \quad \text{on} \quad s^* = 1 \]

\[ \therefore \text{solution is } T^* = \sum_{n=1}^{\infty} B_n (s^* - \frac{1}{s^n}) g_n (\cos \phi) \]

as general soln

This must match with outer solution as \( s^* \to \infty \) and \( \theta \to 0 \), \( r^* \to 1 \)

The inner & outer coordinates are related by:

\[ \varepsilon s^* \cos \phi = r^* \cos \theta - 1, \]
\[ \varepsilon s^* \sin \phi = r^* \sin \theta \]

As \( r^* \to 1 \), outer solution has the form

\[ T^* = \frac{1}{r^*} (r^* - 1) \cos \theta = \frac{1}{1 - (1 - r^*)} \left( (1 - (1 - r^*))^2 - 1 \right) \cos \theta \]

\[ \varepsilon = 2(1 - r^*) \cos \theta \]

and as \( \theta \to 0 \),

\[ T^* \to 2(1 - r^*) \]
Thus limit of inner solution in outer region is, to leading order,

\[ T^* \rightarrow B_1 s^* \cos \phi \rightarrow B_1 \frac{(r^* \cos \Theta - 1)}{\varepsilon} = \frac{B_1}{\varepsilon} (r^* - 1) \]

hence to match solutions to leading order,

\[ B_1 = 2 \varepsilon, \quad B_n = 0 \quad n \neq 1 \Rightarrow \text{these terms would give rise to } O(r^* n) \text{ terms as } r^* \rightarrow \infty, \text{ thus cannot match!} \]

and we have the inner solution:

\[ T^* = 2 \varepsilon (s^* - \frac{1}{B_1}) \cos \phi \]

and outer solution:

\[ T^* = (r^* - \frac{1}{r^*}) \mu \]
Forced Convection Past a Heated Sphere

=> classic problem, illustrates technique of matched asymptotic expansion

Ref: Acrivos & Taylor, Phys. Fluids 5, 387 (1962)

\[
\begin{align*}
\nabla \cdot T & = T - T_0, \\
U & = \frac{T_0}{T_0 - T}, \ n
\end{align*}
\]

Know that \( T^* = \frac{T - T_0}{T_0 - T_0} = f^*(Re, Pr) \)

from dimensional analysis.

Look at case where \( Re = 0 \) (creeping flow)

we thus know solution to eqn of motion \( \Rightarrow \) just stoke's flow past sphere. Left w/:

\[
\nabla^2 T^* = Pe \nabla^2 \cdot T^*
\]

where \( Pe = \frac{\text{conv.}}{\text{cond.}} = \frac{Ua}{\alpha} = Re Pr \)

Can have \( Re \ll 1 \) & \( Pe = O(1) \) or even \( \gg 1 \) as \( Pr \) very large for viscous fluids but look at limit:

\( Re \ll Re Pr \ll 1 \)

Let \( Pe = \epsilon \ll 1 \) (slow flow)
have B.C.'s: $T^* = 1$ at $r^* = 1$,
$T^* = 0$ as $r^* \to \infty$

problem has axial symmetry => use spherical-polar coordinates

$\rho, \mu \equiv \cos \theta$ -- weas. from rear stagnation point

\[
\nabla^2 T = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \mu} \left[ (1-\mu^2) \frac{\partial T}{\partial \mu} \right]
\]

\[
= \varepsilon \left\{ \left( 1 - \frac{3}{2r^2} + \frac{1}{2r^3} \right) \mu \frac{\partial T}{\partial \mu} + \frac{1-\mu^2}{\mu} \left( 1 - \frac{3}{2r^2} - \frac{1}{4r^3} \right) \frac{\partial T}{\partial \mu} \right\}
\]

where we have substituted the velocity profile for creeping flow past a sphere.

Now when $\varepsilon = 0$, assume that convection is negligible & obtain solution (harmonic)

\[ \nabla^2 T = 0 \quad \Rightarrow \quad T_0 = \frac{1}{\nu} \quad \text{(satisfies B.C.'s)} \]

what is next order term?

Let \[ T = \frac{1}{\nu} + \varepsilon T_1 + \cdots \]
Then:\n\[ \nabla^2 T_1 = -\frac{\mu}{r^2} \left( 1 - \frac{3}{2n} + \frac{1}{2r^3} \right) \sim O(\varepsilon) \text{ problem} \]

which has the particular solution:\n\[ T_1^p = \left( \frac{1}{2} - \frac{3}{4n} - \frac{1}{8r^3} \right) \mu \]

but 1st term is just \( \frac{1}{2} \mu \) at \( \infty \).

Can't kill off \( \mu \) harmonics \( \rightarrow \) term which is \( \text{cost at } \infty \) not \( \text{prop. to } \mu \).

Won't work!

Problem: leading order soln was not uniformly valid approx. at \( \infty \) \( \Rightarrow \) just like flow past a sphere.

Far from sphere, temp distib. governed by:\n\[ \varepsilon U_\infty \nabla T = \nabla^2 T \] 

terms balance for \( n \sim O(\frac{1}{\varepsilon}) \).

How do we solve problem? Divide into two regions:
\[ 0 < r \leq O(\frac{1}{\varepsilon}), \quad r \geq O(\frac{1}{\varepsilon}) \]
Then match solns for $\nu = O(\frac{1}{\varepsilon}) \Rightarrow$ region of overlap: method of matched asymptotic expansions.

*Inner Region:*

Let $T(r, \mu, \varepsilon) = \frac{1}{\nu} + \sum_{n=1}^{\infty} f_n(\varepsilon) T_n(r, \mu)$

where $f_1 \to 0$ as $\varepsilon \to 0$ & $f_{n+1} \to 0$ as $\varepsilon \to 0$

we suppose (justify later) $f_1 \equiv \varepsilon$

\[ T_1 = \left( \frac{1}{2} - \frac{3}{4\pi} - \frac{1}{8\pi^3} \right) \mu + \sum_{n=0}^{\infty} \left( A_n r^n + B_n r^{-n+1} \right) P_n(\mu) \]

\[ \Rightarrow \text{particular solns} \quad \Rightarrow \text{homogeneous solns} \]

where $P_n(\mu) \equiv$ Legendre polynomials

$P_0 = 1, \ P_1 = \mu, \ P_2 = \frac{1}{2} (3\mu^2 - 1), \ldots$

We have B.C. at $r = 1$ in inner region:

$T_1 = 0$ at $r = 1$ (i.e., $T_1 = 1 + f_1 T_1 + \ldots = 1$)

Thus $A_n = -B_n$, $n = 0, 2, 3, \ldots$

& $A_1 + B_1 = \frac{3}{8}$
we may determine $B_n$ from matching condition w/ outer region.

**Outer sol’n:**

choose new length scale.

Let $\xi = \epsilon \eta$, i.e. $\xi = O(1)$ in region of overlap.

Hence

$$T^0(\xi, \eta, \epsilon) =$$

$$\nabla_\eta^2 T^0 = \frac{1}{2} \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial T^0}{\partial \xi} \right) + \frac{1}{\xi} \frac{\partial}{\partial \eta} \left[ (1-\epsilon^2) \frac{\partial T^0}{\partial \eta} \right]$$

$$= \left( 1 - \frac{3 \epsilon}{2 \xi^2} + \frac{\epsilon^3}{25} \right) \mu \frac{\partial T^0}{\partial \xi} + \frac{1-\epsilon^2}{\xi} \left( 1 - \frac{\epsilon^3}{4 \xi^2} + \frac{\epsilon^3}{45} \right) \frac{\partial T^0}{\partial \eta}$$

Velocity profile in terms of $\xi$.

Let $T^0 = \sum_{n=0}^{\infty} F_n T^0_n(\xi, \eta)$

i.e. to leading order simply set $\epsilon = 0$ in eq. 4.

Thus:

$$\nabla_\eta^2 T^0 = \frac{\mu}{\xi} \frac{\partial T^0}{\partial \xi} + \frac{1-\epsilon^2}{\xi} \frac{\partial T^0}{\partial \eta}$$

(just eq’n w/ free stream velocity).
The solution to this problem which satisfies B.C. at $z = 0$ is:

$$F_0(\varepsilon) T_0^0(z, \mu) = \frac{F_0(\varepsilon)}{8} e^{-\frac{z}{2}(1-\mu)} \sum_{n=0}^{\infty} C_n P_n(\mu) \sum_{m=0}^{\infty} \frac{(n+m)!}{(n-m)! m!} z^{-m}$$

We require this to match with inner solution:

$$T = \frac{1}{\nu} + \varepsilon \left( \frac{1}{2} - \frac{3}{4r} - \frac{1}{8r^3} \right) \nu + \sum_{n=0}^{\infty} \left\{ A_n \varepsilon^n + B_n \varepsilon^{-n} \right\} a_n$$

as $r \to O(\frac{1}{\varepsilon})$ and $z \to O(\varepsilon)$

Rewriting outer solution in inner variables:

$$F_0(\varepsilon) T_0^0(z, \mu) = \frac{F_0(\varepsilon)}{\varepsilon^{1/2}} e^{-\frac{1}{2}(1-\mu)} \sum_{n=0}^{\infty} C_n P_n(\mu) \sum_{m=0}^{\infty} \frac{(n+m)!}{(n-m)! m!} \left( \frac{1}{\varepsilon} \right)^{-m}$$

This will only match leading term of inner solution $(*)$ if $C_n = 0$ for $n \neq 0$, $C_0 = 1$ and $F_0(\varepsilon) = \varepsilon$

Thus outer solution becomes:

$$T_0^0(z, \mu) \approx \frac{1}{\nu} e^{-\frac{z}{2}(1-\mu)} + O(F, \varepsilon)$$

$$\approx \frac{1}{\nu} \left[ 1 - \frac{1}{2}(1-\mu) + \ldots \right] + O(F, \varepsilon)$$

which must match inner solution:

$$T \approx \frac{1}{\nu} + \frac{1}{2} \varepsilon \mu + \sum_{n=0}^{\infty} A_n \varepsilon^n P_n(\mu) \text{ for large } \nu$$
Thus we see that $\frac{\varepsilon}{2} \mu$ term matches automatically. Also, matching only works if $A_n = 0 \; n \neq 0$ and $A_0 = -\frac{1}{2}$.

Thus in the inner region:

$$T = \frac{1}{\nu} + \varepsilon \left\{ \frac{1}{2} \left( \frac{1}{\nu} - 1 \right) + \mu \left( \frac{1}{2} - \frac{3}{4\nu^2} + \frac{3}{8\nu^2} - \frac{1}{8\nu^3} \right) \right\} + O(f_2)$$

$$\frac{Q}{4\pi \alpha K (T_0 - T_\infty)} = \frac{1}{2} \int_{-1}^{1} \left( \frac{d^2 T}{d\xi^2} \right) d\mu$$

$Q =$ total rate of ht. transf.

Obtain $\text{Nu} = 1 + \frac{\varepsilon}{2} + O(f_2)$.

Interesting to note that $f_2 \neq \varepsilon^2$ rather obtain term $\varepsilon^2 \log \varepsilon$ which decays away more slowly.

Result above can be generalized to bodies of arbitrary shape, e.g.:

$$\text{Nu} = \text{Nu} \bigg|_{\varepsilon = 0} (1 + \frac{\varepsilon}{2})$$

because far-field is shape-indep.