Wall Reflections in Stokes Flow

Stokes flow = Zero Re

If there is no inertia the g's are linear

Reversal in Time = Reversal in Dir.

Look at a sphere near a plane in simple shear:

By reversibility, there is no net displacement away from the wall.

What if it isn't a sphere?

May get displacement but
It's reversible! Particle would rotate (Jeffrey's Orbit) and there is no net displacement at the end of a period!

center just oscillates!

What about a drop? (or RBC) - deformation breaks linearity, yields a net drift!

How do we evaluate? Simpler method: look at far-field reflection due to the wall!
Simplest example: Reflection of a Stokeslet (point force) 

1. sphere settling toward wall

- $h$

- $h$

2. image point

The image of a sphere in a stress-free surface is just a sphere moving with opposite velocity!

Simplest problem: a sphere of radius $a$ & $\Delta g$ rising toward a free surface. How much does it slow down?

$$F_i = \frac{4}{3} \pi \Delta g a^3 \Delta t$$
To get the velocity, we calculate the velocity in the presence of a sphere with opposite sign (as) at the image point and evaluate it back at the sphere!

\[ F_i \]

\[ \hat{F}_i' \text{ (reflection)} \]

The velocity due to the reflection is: (far-field)

\[ u_i' = \frac{1}{8\pi\mu} \left( \frac{\delta_i}{R} + \frac{(x_i - \hat{x}_i)(x_i - \hat{y}_i)}{R^3} \right) F_i' \]

where \( \hat{x}_i \) is the image location

\( R = |x_i - \hat{x}_i| \)

We evaluate back at the original location \( y_i \).
Let's take $y_i = h S_i^3$

So:

$$u_i = \frac{1}{\pi \mu} \left( \frac{8i i}{2h} + \frac{2h S_i^3}{(2h)^3} \right)$$

$$= -\frac{4}{3} \pi \alpha a^3 \left( \frac{S_i^3}{2h} + \frac{S_i^3}{2h} \right)$$

$$= -\frac{1}{6} \frac{\alpha a^3}{\mu} \left( \frac{a}{h} \right)$$

The velocity relative to this is:

$$\frac{4}{3} \pi a^3 \frac{\alpha a^3}{6 \pi \mu a} = \frac{2}{9} \frac{\alpha a^2}{\mu}$$

Stokes law

So the net velocity is:

$$u_i = \frac{2}{9} \frac{\alpha a^2}{\mu} \left( 1 - \frac{3a}{4h} + O\left(\frac{a^2}{h^2}\right) \right) x S_i^3$$
This is actually pretty good right up to \( a/h \approx 1 \) (it can't get any closer.)

What about a rigid wall? You do this the same way, but the image system is more complex!

Look at a Stokeslet. This was worked out by Blake (1971) for a normal Stokeslet the image is a Stokeslet of opposite sign and the combination of a Stokes doublet (stresslet) and a source doublet.

Much messier!

If you evaluate it, you get

\[
u_i = \frac{z}{9} \frac{\partial \delta_i}{\partial \xi} \delta_i \left( 1 - \frac{9}{8} \frac{a}{h} + O\left( \frac{a^2}{h^2} \right) \right)
\]
The image system in this case consists of a stokeslet of equal magnitude and opposite in sign (i.e. in $x_1$ or $x_3$-direction), a stokes-doublet of strength $2\lambda$ of the stokeslet and a source-doublet of strength $2\lambda^2$ of the stokeslet. The stokes-doublet consists of equal and opposite stokeslets orientated in the normal ($x_3$) direction to the plane boundary, and with their displacement axis in the $x_1$ (or $x_3$) direction which in the tensorial notation is represented by $D_{31}$ (or $D_{33}$). Similarly, the source-doublet has displacement axis in the $x_3$-direction. The stokes-doublet in this case is of magnitude $D_{33} = 2\lambda F_1$, which is equal in strength in the far-field to the combined influence of the initial and image stokeslets.

Fig. 3. Diagram illustrating image system for $k = 1$, the strength of the components being given in brackets.

Far-field: Stokes-doublet

![Image system: Stokeslet $(-F)$ Stokes-doublet $(2\lambda F)$ Source-doublet $(-2\lambda^2 F)$]

Fig. 4. Diagram illustrating image system for $k = 3$, the strength of the components being given in brackets.
Or, what about the image system of a stresslet?

Any force & torque free particle has as its far field a stresslet!

This is described by the symmetric, traceless second-order tensor $S_{ijk}$

The velocity due to $S_{ijk}$ is:

$$u_i = -\frac{3}{2} \frac{S_{ijk}}{4\pi \mu \omega} \frac{x_i x_j x_k}{r^5}$$

We can construct any stresslet by a combination of Stokeslets!

For example: (Stokes doublet)
\[ F_i = \frac{F}{a} \delta_{ii} \quad \text{for} \quad i = 1, 2, \ldots, n \]

\[ F_i = -\frac{F}{a} \delta_{ii} \]

\[ S_{ij} = \frac{1}{3} F \left( 3 \delta_{ii} \delta_{jj} - \delta_{ij} \right) \]

Figure 3.1. A simple stresslet.
This suggests a simple way to get the image velocity: just build up the image system of the component Stokeslets from Blake!

If we do this, and evaluate it at the original location $x_3 = h$ we get:

$$u'_i = -\frac{1}{8\pi \mu} \frac{3}{4h^2} S_{ij} \left( \delta_{ij} + \frac{1}{2} \delta_{ij} \delta_{jj} \right)$$

$x_3 = h$
$x_1 = x_2 = 0$

In particular,

$$u'_3 = -\frac{1}{8\pi \mu} \frac{9}{8h^2} S_{33}$$

$x_3 = h$

So the velocity normal to a plane is just proportional to $S_{33}$.!
What happens for a sphere?

for the shear flow \( u_i' = \delta \times \delta_{i1} \)

we have the stress \( \tau_{ij} \):

\[
\tau_{ij} = \frac{10 \pi}{3} \mu \delta a^3 (\delta_{i1} \delta_{j3} + \delta_{i3} \delta_{j1})
\]

Thus, \( \tau_{33} \) is zero! There's no drift toward or away from the plane (as expected)

What about \( u_{i1}' \)?

\[
u_{i1}' = u_{i1} \delta_{i1} = -\frac{1}{8 \pi \mu a} \frac{3}{4 h^2} s_{13} (1 + \frac{1}{2})
\]

\[
= -\frac{1}{8 \pi \mu} \frac{9}{8 h^2} s_{13}
\]

where \( s_{13} = \frac{10 \pi}{3} \mu \delta a^3 \)

so \( u_{i1}' = -15 \frac{\delta a^3}{32 \ h^2} \) so it slows down a bit.
What happens for a drop?
The stresslet depends on the capillary number and viscosity ratio:

\[ \text{Ca} = \frac{\gamma \mu a}{\mu} \quad \chi = \frac{\mu_{\text{drop}}}{\mu_{\text{fluid}}} \]

\[ S = \frac{(8\mu)(4\pi a^3)\text{Ca}}{(3\mu)(8\pi a^3)\text{Ca}} \left( \frac{19\lambda + 16}{80(\lambda + 1)^2} \right) \frac{25\lambda^2 + 41\lambda + 4}{7(\lambda + 1)} \]

\[ \approx 8\mu \left( \frac{4\pi a^3}{3} \right) \text{Ca} (-3.11) \]

\( \Rightarrow \chi = 0.083 \) in our exps...

This yields a normal velocity of:

\[ u_3 = (-3.11) \frac{8\mu a^3}{8\mu a^3} \left( \frac{4\pi a^3}{8\mu a^3} \right) \left( \frac{-1}{8\mu a^3} \right) \frac{9}{8h^2} \]

\[ = (3.11) \left( \frac{3}{16} \right) \frac{8a^3}{h^2} \text{Ca} \]

You get a similar expression for drift from a free surface.
Note that the slopes of the plots of \((h/a)^3\) versus dimensionless time increase as the capillary number increases. This is as expected: since the droplets are more deformed at higher capillary numbers, the drift velocity should increase. In plotting the slopes of these lines as a function of capillary number (Fig. 6), we find that the drift velocity increases linearly in capillary number, as expected from the order capillary number deformation theory. The vertical line in Fig. 6 is the capillary number at which droplet breakup occurred, and the inclined line is the theoretical prediction, which will be discussed in the next section.

The expectation that \((h/a)^3\) grows linearly in time is obtained by only considering the contribution of the first reflection to the drift velocity of the droplet. This will be strictly valid only as \(h/a\) becomes large. For small values of \(h/a\) it is necessary to consider higher-order reflections. We may obtain information about these higher-order terms by examining the drift velocity for small values of \(h/a\). In Fig. 7 we have plotted the observed trajectory together with the limiting far-field linear trajectory for several capillary numbers for drift normal to the outer Couette wall. Note that for the highest capillary numbers (close to droplet breakup) the deviation from the linear relationship is quite large and persists to values of \(h/a\approx8\). In contrast, at smaller capillary numbers the linear relationship is preserved down to \(h/a\approx2\). In all cases the far field was used to calculate the drift velocity used in Fig. 6.

We have also neglected the contribution of the second wall of the Couette device in calculating the drift velocity. Because of the inverse square dependence of the drift velocity on distance from the wall, we expect the contribution of the second wall relative to the first to be of \(O(h^2/(d-h)^2)\), where \(d\) is the gap width; thus, the second wall may be neglected provided this parameter is small. In our experiments, however, we found that the linear relationship was preserved out to values of \(h^2/(d-h)^2\) as great as 0.4 (Fig. 4), although the larger droplets would eventually achieve some steady position as was described earlier. The velocities used in Fig. 6 were estimated from droplet positions in the outer third of the Couette gap.

3. Drift normal to a free surface

Droplets were found to drift away from both the upper and lower interfaces. We conducted drift measurements only normal to the upper air–castor oil interface, however, since the interfacial shear stresses are much lower here than at the mercury–castor oil interface. In Fig. 8 we plot the drift
ok, what about wall flipping? We used this earlier, now we look at where it comes from!

The key is that the disturbance (reflection) velocity "sucks down" fluid approaching from above, and "pushes up" fluid approaching from below.

This means that some fluid elements never get close to a sphere!

[Diagram showing fluid flow around a sphere with zero velocity streamline relative to sphere]
Figure 2.3. Stresslet reflection due to the presence of a wall. The image point is located on the opposite side of the wall at the same distance $h$ from the wall as the original stresslet.
The image system for a sphere stresslet was evaluated by Ankit Rohatgi, and at the plane of the sphere \( x_3 = h \) it yields:

\[
  u'_i = \frac{5}{2} a^3 \dot{\gamma} \left[ \frac{-(x_3 + 2h)(x_1^2 + 2h(x_3 + h))}{R^5} + \frac{10(x_3 + h)(x_3 + 2h)x_1^2 h}{R^7} \right]
\]

\[
  u'_2 = \frac{5}{2} a^3 \dot{\gamma} \left[ \frac{-x_1 x_2 (x_3 + 2h)}{R^5} + \frac{10x_1 x_2 h (x_3 + h)(x_3 + 2h)}{R^7} \right]
\]

\[
  u'_3 = \frac{5}{2} a^3 \dot{\gamma} \left[ \frac{-x_1 (x_3^2 + 4x_3 h + 2h^2)}{R^5} + \frac{10x_1 h (x_3 + h)(x_3 + 2h)^2}{R^7} \right]
\]

This is combined with the velocity profile

\[
  u^s_i = \dot{\gamma} x_3 \delta_{i1} - \frac{\dot{\gamma}}{2} (x_3 \delta_{i1} + x_1 \delta_{i3}) \frac{a^5}{r^5} - \frac{5}{2} \dot{\gamma} x_1 x_3 x_i \left( \frac{a^3}{r^5} - \frac{a^5}{r^7} \right)
\]

\[
  \uparrow \text{undisturbed flow} \quad \uparrow\text{stresslet} \quad \sim \frac{1}{r^2} \text{far-field}
\]

The combined fields can be integrated:
Figure 2.4. In-plane ($y/a = 0$) fluid streamlines for a simple shear flow past a sphere (top) and for a sphere located at $h/a = 6$ from the lower wall (bottom). The open-loop flipping trajectories appear in the presence of the wall.
Both the fluid and self-diffusivity may be connected to the random walk motions via the simple integral given by da Cunha and Hinch (1996)

\[
D'_i = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\Delta X_i)^2 n\phi |z^{-\infty}|dz^{-\infty}dy^{-\infty}
\]

(2.4)

where \(\Delta X_i\) is the displacement of a tracer resulting from a particular interaction trajectory produced by a suspension sphere approaching from the initial location \((\pm\infty, y^{-\infty}, z^{-\infty})\) and \(n\) is the number density of interacting spheres. It can be

Using da Cunha & Hinch's method for calculating the self-diffusivity given in Equation (2.4), we can determine the random walk fluid dispersivity in the gradient direction. Integrating over the envelope of flipping trajectories, the fluid gradient diffusivity is given by

\[
D'_3 = \frac{2}{3} \frac{\phi \dot{\gamma}}{\pi a^3} \int_0^{2\sqrt{2/\gamma}} \int_0^{\Delta z_{lim}} (2z)^2 z dz dy
\]

(2.20)

which yields

\[
D'_3 = \frac{25}{3\pi} \left\{ \frac{1441\sqrt{2} + 3321\tan^{-1} \sqrt{2}}{55296} \right\} \dot{\gamma} a^3 \phi = 0.24995 \dot{\gamma} a^3 \phi \approx \frac{1}{4} \dot{\gamma} a^3 \phi.
\]

(2.21)