CBE 60545

Transport Phenomena II

Course Notes

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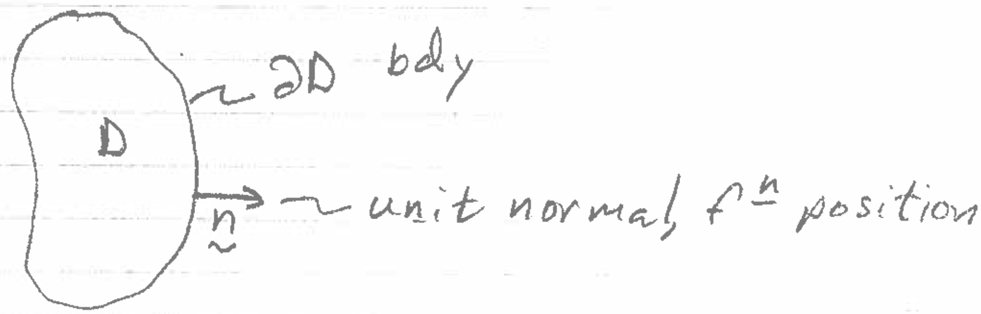
Deriv. Transport Eq'ns:

Review mat'l in ch 8 BSdL ⇒ don't memorize, but know it exists & how to use. Deals w/ ~~[illegible] Law~~ fourier law of ht. cond. & thermal conductivity

Consider arbitrary volume D:



We have some fluid flowing thru volume

Derive transp. eq'ns from conservation laws:

Mass is conserved, thus

$$\left\{\begin{matrix}\text{rate of}\\\text{accumulation}\end{matrix}\right\} = -\left\{\begin{matrix}\text{net rate outflow}\\\text{by convection}\end{matrix}\right\}$$

Now rate of accumulation is simply

$$\{\text{acc.}\} = \frac{d}{dt}\int_D \rho \, dV = \int_D \frac{\partial \rho}{\partial t} \, dV$$

And rate of outflow *must* occur at surface ∂D. Flux thru any surface is just

$$\text{flux} = \rho(\underset{\sim}{u} \cdot \underset{\sim}{n})$$ ~ component of velocity in dir. of unit normal

Thus

$$\begin{Bmatrix} \text{net rate outflow} \\ \text{by convection} \end{Bmatrix} = \int_{\partial D} \rho \underset{\sim}{u} \cdot \underset{\sim}{n} \, dA$$

or
$$\int_D \frac{\partial \rho}{\partial t} \, dV + \int_{\partial D} \rho \underset{\sim}{u} \cdot \underset{\sim}{n} \, dA = 0$$

Now use Divergence Theorem:

$$\int_{\partial D} \underset{\sim}{f} \cdot \underset{\sim}{n} \, dA = \int_D \underset{\sim}{\nabla} \cdot \underset{\sim}{f} \, dV$$

Thus

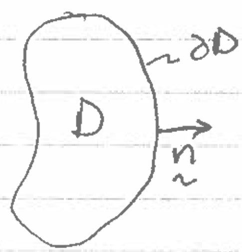
$$\int_D \left[\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u}) \right] dV = 0$$

Since D is <u>arbitrary</u>, eq'n valid at all pts, thus:

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u}) = 0$$

which is eq'n of continuity. Will use same mass balance for multicomponent case to derive mass transp. eq'ns later.

Now look at energy; also conserved quantity!



$\left\{ \begin{array}{c} \text{Rate of} \\ \text{accumulation} \\ \text{of energy} \end{array} \right\} = - \left\{ \begin{array}{c} \text{net rate of energy } \underline{\text{out}} \\ \text{by convection} \end{array} \right\}$

$- \left\{ \begin{array}{c} \text{net rate } \underline{\text{out}} \\ \text{by conduction} \end{array} \right\} - \left\{ \begin{array}{c} \text{net rate work } \underline{\text{by}} \\ \text{system } \underline{\text{on}} \text{ surroundings} \\ \text{(we include gravity} \\ \text{work here)} \end{array} \right\}$

$+ \left\{ \text{sources} \right\}$ ignore for now

The energy of fluid element is just internal energy + K.E. (pot. E included as work term), thus

Tot. E. per unit <u>volume</u> $\equiv \rho \hat{U} + \frac{1}{2} \rho u^2$ (int. E/unit mass)

Thus:

$$\left\{\begin{matrix}\text{Rate of}\\ \text{acc. of } E\end{matrix}\right\} = \int_D \frac{\partial}{\partial t}\left(\rho\hat{U} + \tfrac{1}{2}\rho u^2\right) dV$$

and

$$\left\{\begin{matrix}\text{net rate } E \text{ out}\\ \text{by convection}\end{matrix}\right\} = \int_{\partial D} \left(\rho\hat{U} + \tfrac{1}{2}\rho u^2\right) \underset{\sim}{u}\cdot\underset{\sim}{n}\, dA$$

let $\underset{\sim}{q}$ be <u>conduction</u> <u>flux</u> <u>vector</u>,

then

$$\left\{\begin{matrix}\text{energy } \underline{\text{out}} \text{ by}\\ \text{conduction}\end{matrix}\right\} = \int_{\partial D} \underset{\sim}{q}\cdot\underset{\sim}{n}\, dA$$

Left w/ work terms.

Suppose surroundings exert force/Area $\underset{\sim}{f}$ <u>on</u> system at surface ∂D

∴ rate of work/area done <u>by</u> system <u>on</u> surroundings

$= -\underset{\sim}{u}\cdot\underset{\sim}{f}$

what is $\underset{\sim}{f}$?

related to stress tensor $\underset{\approx}{\sigma}$

$\sigma_{ij} \equiv F/A$ exerted by fluid of greater i on fluid of lesser i in j direction

Thus, $\underset{\sim}{f} = \underset{\approx}{\sigma} \cdot \underset{\sim}{n}$!

Now recall that $\underset{\approx}{\sigma} = -p\,\underset{\approx}{I} + \underset{\approx}{\tau}$

identity matrix

dynamic or viscous stress

(in Einstein notation,

$$\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij} \;)$$

Note: BS&L uses opposite sign for τ_{ij} $\Rightarrow$ don't worry about it, just keep definition straight.

So: work/Area by system on surroundings is:

$$W/A = -\underset{\sim}{u} \cdot \underset{\sim}{f} = -\underset{\sim}{u} \cdot (\underset{\approx}{\sigma} \cdot \underset{\sim}{n})$$

$$= -\underset{\sim}{u} \cdot (-p\,\underset{\approx}{I} + \underset{\approx}{\tau}) \cdot \underset{\sim}{n}$$

$$= +p\,\underset{\sim}{u} \cdot \underset{\sim}{n} - \underset{\sim}{u} \cdot \underset{\approx}{\tau} \cdot \underset{\sim}{n}$$

pressure work

viscous work

We also have work done against gravity, given by:

$$\left\{ \begin{matrix} \text{rate of work done} \\ \text{\underline{by} gravity \underline{on}} \\ \text{system} \end{matrix} \right\} = \underbrace{\int_D \rho \underset{\sim}{u} \cdot \underset{\sim}{g} \, dV}_{\text{source of K.E.}}$$

Putting it all together:

$$\underbrace{\int_D \frac{\partial}{\partial t}\left(\rho \hat{U} + \tfrac{1}{2}\rho u^2\right) dV}_{\text{Accumulation}} = \underbrace{- \int_{\partial D} \left(\rho \hat{U} + \tfrac{1}{2}\rho u^2\right) \underset{\sim}{u} \cdot \underset{\sim}{n} \, dA}_{\text{Convection}}$$

$$\underbrace{- \int_{\partial D} \underset{\sim}{q} \cdot \underset{\sim}{n} \, dA}_{\text{conduction}} \underbrace{- \int_{\partial D} p \underset{\sim}{u} \cdot \underset{\sim}{n} \, dA}_{\text{pressure work}}$$

$$\underbrace{+ \int_{\partial D} \underset{\sim}{u} \cdot \underset{\approx}{\tau} \cdot \underset{\sim}{n} \, dA}_{\text{viscous work}} \underbrace{+ \int_D \rho \underset{\sim}{u} \cdot \underset{\sim}{g} \, dV}_{\text{gravity work}}$$

Applying the Divergence thm to RHS:

$$\int_D \frac{\partial}{\partial t}\left(\rho \hat{U} + \tfrac{1}{2}\rho u^2\right) dV = -\int_D \underset{\sim}{\nabla}\cdot\left[\left(\rho \hat{U} + \tfrac{1}{2}\rho u^2\right)\underset{\sim}{u}\right] dV$$

$$-\int_D \underset{\sim}{\nabla}\cdot \underset{\sim}{q}\, dV - \int_D \underset{\sim}{\nabla}\cdot\left(p\underset{\sim}{u}\right) dV$$

$$+\int_D \underset{\sim}{\nabla}\cdot\left(\underset{\approx}{\tau}\cdot\underset{\sim}{u}\right) dV + \int_D \rho\underset{\sim}{u}\cdot\underset{\sim}{g}\, dV$$

Or, since D is arbitrary,

$$\frac{\partial}{\partial t}\left(\rho \hat{U} + \tfrac{1}{2}\rho u^2\right) = -\underset{\sim}{\nabla}\cdot\left[\left(\rho \hat{U} + \tfrac{1}{2}\rho u^2\right)\underset{\sim}{u}\right] - \underset{\sim}{\nabla}\cdot\underset{\sim}{q}$$

$$-\underset{\sim}{\nabla}\cdot\left(p\underset{\sim}{u}\right) + \underset{\sim}{\nabla}\cdot\left(\underset{\approx}{\tau}\cdot\underset{\sim}{u}\right) + \rho\underset{\sim}{u}\cdot\underset{\sim}{g}$$

Still a bit unweildy.

Now diff. first term on RHS by parts:

$$\underset{\sim}{\nabla}\cdot\left[\left(\rho \hat{U} + \tfrac{1}{2}\rho u^2\right)\underset{\sim}{u}\right] = \rho\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\left(\hat{U} + \tfrac{1}{2}u^2\right)$$

$$+\left(\hat{U} + \tfrac{1}{2}u^2\right)\underset{\sim}{\nabla}\cdot\left(\rho\underset{\sim}{u}\right)$$

And term on LHS:

$$\frac{\partial}{\partial t}\left(\rho\hat{U} + \frac{1}{2}\rho u^2\right) = \rho\frac{\partial}{\partial t}\left(\hat{U} + \frac{1}{2}u^2\right) + \left(\hat{U} + \frac{1}{2}u^2\right)\frac{\partial \rho}{\partial t}$$

Thus we obtain:

$$\rho\left(\frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\right)\left(\hat{U} + \frac{1}{2}u^2\right) \equiv \rho\frac{D}{Dt}\left(\hat{U} + \frac{1}{2}u^2\right)$$

$$= -\left(\hat{U} + \frac{1}{2}u^2\right)\underbrace{\cancel{\left[\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}\cdot(\rho\underset{\sim}{u})\right]}}_{0 \text{ from C.E.}}$$

$$-\underset{\sim}{\nabla}\cdot\underset{\sim}{q} + \rho\underset{\sim}{u}\cdot\underset{\sim}{g} - \underset{\sim}{\nabla}\cdot(P\underset{\sim}{u}) + \underset{\sim}{\nabla}\cdot(\underset{\approx}{\tau}\cdot\underset{\sim}{u})$$

which is the equation of energy from viewpoint of an observer moving <u>with</u> the fluid ⇒ Lagrangian viewpoint.

Still a bit unweildy ⇒ remove Mech. E terms.

Mech. Energy Balance ⇒ derivable from N-S eq'ns.

Recall:

$$\rho\frac{D\underset{\sim}{u}}{Dt} = -\underset{\sim}{\nabla}P + \rho\underset{\sim}{g} + \underset{\sim}{\nabla}\cdot\underset{\approx}{\tau}$$

note sign dif. from BS&L

Dot w/ velocity & obtain:

$$\rho \frac{D(\frac{1}{2}u^2)}{Dt} = -\underset{\sim}{u}\cdot\underset{\sim}{\nabla} P + \left(\underset{\sim}{u}\cdot(\underset{\sim}{\nabla}\cdot\underset{\approx}{\tau})\right) + \rho\,\underset{\sim}{u}\cdot\underset{\sim}{g}$$

$$= p(\underset{\sim}{\nabla}\cdot\underset{\sim}{u}) - \underset{\sim}{\nabla}\cdot(p\underset{\sim}{u}) \qquad\qquad = \underset{\sim}{\nabla}\cdot(\underset{\approx}{\tau}\cdot\underset{\sim}{u}) - \underset{\approx}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{u}$$

Thus we obtain:

Total pressure work

$$\rho \frac{D(\frac{u^2}{2})}{Dt} = p(\underset{\sim}{\nabla}\cdot\underset{\sim}{u}) - \underset{\sim}{\nabla}\cdot(p\underset{\sim}{u}) + \rho\,\underset{\sim}{u}\cdot\underset{\sim}{g}$$

↳ Rev. conv. of KE to IE

$$+\, \underset{\sim}{\nabla}\cdot(\underset{\approx}{\tau}\cdot\underset{\sim}{u}) - \underset{\approx}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{u}$$

↳ irrev. conversion of mech. E to thermal E

total viscous work

Subtraction from total E eq'n yields:

$$\rho\frac{D\hat{U}}{Dt} = -\underset{\sim}{\nabla}\cdot\underset{\sim}{q} - P(\underset{\sim}{\nabla}\cdot\underset{\sim}{u}) + \underset{\approx}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{u}$$

↳ may be pos. or neg.

End lec. 1

Thermal energy eq'n

More convenient to work w/ eq'n in terms of temp.

From thermo, assuming no changes in composition,

$$U = f^{n}(\hat{V}, T)$$

$$\therefore\ d\hat{U} = \left(\frac{\partial \hat{U}}{\partial \hat{V}}\right)\Bigg|_T d\hat{V} + \left(\frac{\partial \hat{U}}{\partial T}\right)_{\hat{V}} dT$$

$$\left(\frac{\partial \hat{U}}{\partial \hat{V}}\right)\Bigg|_T d\hat{V} \to \left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right] d\hat{V} \qquad \left(\frac{\partial \hat{U}}{\partial T}\right)_{\hat{V}} dT \to \hat{C}_V\, dT$$

from thermodynamics

Thus

$$\rho \frac{D\hat{U}}{Dt} = \left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right] \rho \frac{D\hat{V}}{Dt} + \rho \hat{C}_V \frac{DT}{Dt}$$

Now

$$\rho \frac{D\hat{V}}{Dt} = \rho \frac{D(1/\rho)}{Dt} = -\frac{1}{\rho}\frac{D\rho}{Dt} = \underset{\sim}{\nabla} \cdot \underset{\sim}{u}$$

from continuity

Thus

$$\rho \frac{D\hat{U}}{Dt} = \left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right] (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) + \rho \hat{C}_V \frac{DT}{Dt}$$

and thus

$$\rho \hat{C}_V \frac{DT}{Dt} = \underbrace{-\underset{\sim}{\nabla} \cdot \underset{\sim}{q}}_{\text{conduction}} \underbrace{- T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}} \underset{\sim}{\nabla} \cdot \underset{\sim}{u}}_{\text{compression}} + \underbrace{\underset{\approx}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{u}}_{\mu\ \text{dissip.}}$$

This form is useful when dealing with gases!

For an ideal gas $P\hat{V} = RT$

$$\therefore\ T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}} = \frac{TR}{\hat{V}} = P$$

Also, for a Newtonian fluid

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\,\underset{\sim}{\nabla}\cdot\underset{\sim}{u}\,\delta_{ij}\right)$$

thus

$$\underset{\approx}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{u} \equiv \mu\,\phi_v = \tau_{ij}\frac{\partial u_i}{\partial x_j} = \mu\,\frac{1}{2}\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 - \frac{4}{3}\left(\underset{\sim}{\nabla}\cdot\underset{\sim}{u}\right)^2\right]$$

positive definite!

And from Fourier Law of ht. conduction:

$$\underset{\sim}{q} = -K\,\underset{\sim}{\nabla}T$$

K → thermal conductivity

Thus

$$\rho\,\hat{C}_v\,\frac{DT}{Dt} = \underbrace{\underset{\sim}{\nabla}\cdot\left(K\,\underset{\sim}{\nabla}T\right)}_{\text{conduction}} - \underbrace{P\,\underset{\sim}{\nabla}\cdot\underset{\sim}{u}}_{\text{adiabatic compression}} + \mu\,\phi_v$$

an example of where this would be used is in calc. of temp. distrib. in a shock tube

this is starting point for energy transp. in gases → note Φ_v usually negligible
at cst. vol.

For liquids & solids, or gases at cst. pressure a more convenient form may be used

Return to thermal E eq'n:

$$\rho \frac{D\hat{U}}{Dt} = -\underset{\sim}{\nabla} \cdot \underset{\sim}{q} - p(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) + \underset{\approx}{\tau} : \underset{\sim}{\nabla} \underset{\sim}{u}$$

Recall from thermo: $\hat{H} = \hat{U} + p\hat{V} = \hat{U} + \frac{p}{\rho}$

Thus

$$\frac{D\hat{H}}{Dt} = \frac{D\hat{U}}{Dt} + \frac{1}{\rho}\frac{Dp}{Dt} + p\frac{D(1/\rho)}{Dt}$$

$$= \frac{D\hat{U}}{Dt} + \frac{1}{\rho}\frac{Dp}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} \longrightarrow -\rho\, \underset{\sim}{\nabla} \cdot \underset{\sim}{u}$$

from C.E.

Thus

$$\rho \frac{D\hat{U}}{Dt} = \rho \frac{D\hat{H}}{Dt} - \frac{Dp}{Dt} - p\, \underset{\sim}{\nabla} \cdot \underset{\sim}{u}$$

And from internal energy eq'n:

$$\rho \frac{D\hat{H}}{Dt} = -\underset{\sim}{\nabla} \cdot \underset{\sim}{q} + \frac{Dp}{Dt} + \underset{\approx}{\tau} : \underset{\sim}{\nabla} \underset{\sim}{u}$$

Now, again from thermo:

$$d\hat{H} = \left(\frac{\partial \hat{H}}{\partial T}\right)_p dT + \left(\frac{\partial \hat{H}}{\partial P}\right)_T dP$$

$$= \hat{C}_p\, dT + \left[\hat{V} - T\left(\frac{\partial \hat{V}}{\partial T}\right)_p\right] dP$$

thus

$$\rho \frac{D\hat{H}}{Dt} = \rho \hat{C}_p \frac{DT}{Dt} + \frac{DP}{Dt} - \beta T \frac{DP}{Dt}$$

where $\beta \equiv \frac{1}{\hat{V}}\left(\frac{\partial \hat{V}}{\partial T}\right)_p$ ≡ thermal expansion

thus

$$\rho \hat{C}_p \frac{DT}{Dt} = -\underset{\sim}{\nabla} \cdot \underset{\sim}{q} + \beta T \frac{DP}{Dt} + \underset{\approx}{\tau} : \underset{\sim}{\nabla} \underset{\sim}{u}$$

Now if we are at cst. pressure then

$\frac{DP}{Dt} = 0$ and

$$\rho \hat{C}_p \frac{DT}{Dt} = -\underset{\sim}{\nabla} \cdot \underset{\sim}{q} + \underset{\approx}{\tau} : \underset{\sim}{\nabla} \underset{\sim}{u}$$

or, for cst k & Newtonian fluids:

$$\rho \hat{C}_p \frac{DT}{Dt} = k \nabla^2 T + \mu \phi_v$$

small in most cases

for incompressible fluids, $\hat{C}_p = \hat{C}_v$ and both equations are the same.

For solids $\underset{\sim}{u} = 0$ and we obtain

$$\rho \hat{C}_p \frac{\partial T}{\partial t} = K \nabla^2 T \quad (\text{const } K)$$

May also have source terms added to RHS such as electrical dissip, nuclear generation, etc.

would be E/time/volume

Suppose we have some general problem, say flow past nuclear element of strange shape. Equations governing problem are:

$$\text{C.E.:} \quad \frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u}) = 0 \qquad (1)$$

$$\text{N-S:} \quad \rho \frac{D\underset{\sim}{u}}{Dt} = -\underset{\sim}{\nabla} P + (\underset{\sim}{\nabla} \cdot \underset{\approx}{\tau}) + \rho \underset{\sim}{g} \qquad (3)$$

$$\text{Thermal:} \quad \rho \hat{C}_p \frac{DT}{Dt} = -(\underset{\sim}{\nabla} \cdot \underset{\sim}{q}) + \beta T \frac{DP}{Dt} \qquad (1)$$

$$+ (\underset{\approx}{\tau} : \underset{\sim}{\nabla} \underset{\sim}{u}) + \dot{S}_{th}$$

we also must have constitutive relation:

$$\underset{\sim}{q} = f^{\underline{n}}(K, T) \; ; \; \tau = f^{\underline{n}}(\mu, \underset{\sim}{u})$$

and $\mu, \rho, K = f^{\underline{n}}(T, P)$

Thus, obtain series of coupled equations, must be solved w/ appropriate B.C.'s & I.C.'s

Very difficult to solve (even numerically)

All physics (except radiative ht. transf, ~~not covered in course~~) included in eq'ns $\Rightarrow$ simply a matter of solution! Physicist would say stop here, as eng. need sol'n $\Rightarrow$

make approximations to obtain sol'n

Problem: apply physical insight to see what approx. are realistic.

OK, now look at energy transp. in solids

we have:

$$\rho \hat{C}_p \frac{DT}{Dt} = -(\underset{\sim}{\nabla} \cdot \underset{\sim}{q}) + \left(\frac{\partial \ln \hat{V}}{\partial \ln T}\right)_p \frac{DP}{Dt} + (\underset{\approx}{\tau} : \underset{\sim}{\nabla} \underset{\sim}{u}) + \dot{S}_{th}$$

Assume $\frac{DP}{Dt} = 0$ (term is small anyway)

also $\underset{\sim}{u} = 0$ in solid

Substituting Fourier Law of ht cond, obtain

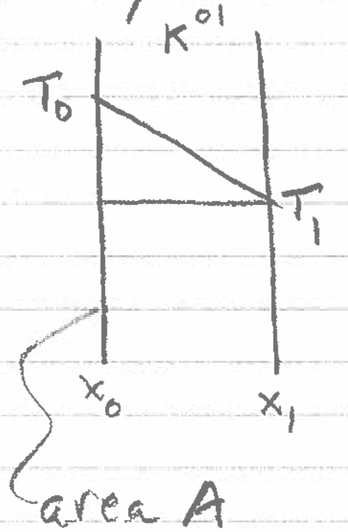
$$\rho \hat{C}_p \frac{\partial T}{\partial t} = \underset{\sim}{\nabla} \cdot (k \underset{\sim}{\nabla} T) + \dot{S}_{th}$$

+ relations of $\rho, \hat{C}_p, k$ to T

Assume $k = cst$

$$\therefore\ \rho \hat{C}_p \frac{\partial T}{\partial t} = k\nabla^2 T + \dot{S}_{th}$$

Now look at conduction thru wall at steady state (insulation problem)



s.s., so $\frac{\partial T}{\partial t} = 0$

Given that outside wall is at T_1, inside wall at T_0, what will be E loss?

(what will be heating cost?)

$\nabla^2 T = 0$, only 1D so $\frac{\partial^2 T}{\partial x^2} = 0$

$\frac{\partial T}{\partial x} = C_1$, $T = C_1 x + C_2$

$T\big|_{(x=x_0)} = T_0$, $T\big|_{x=x_1} = T_1$

$$\therefore\ T = T_0 + \frac{x - x_0}{x_1 - x_0}(T_1 - T_0)$$

linear profile!

As a consequence of geometry, energy flux thru all planes are equal

$$\left(\text{i.e.,}\ \underset{\sim}{\nabla} \cdot \underset{\sim}{q} = 0 \Rightarrow \frac{\partial q_x}{\partial x} = 0 \ \therefore\ q_x = cst \right)$$

$$q_x = -k^{01} \frac{\partial T}{\partial x} = -k^{01} \frac{T_1 - T_0}{x_1 - x_0} = -k \frac{\Delta T}{\Delta x}$$

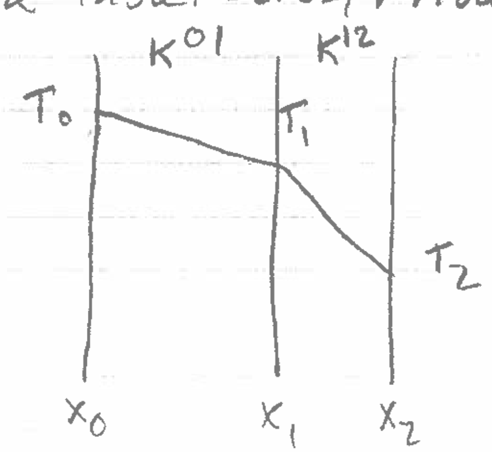
<u>and</u> $\dot{Q} = A q_x \Rightarrow$ total E loss thru wall

Analogy to electrical conduction $\Rightarrow$ voltage drop across resistor <u>exactly</u> equiv. to temp. drop across insulator.

Current equiv. to $\dot{Q}$, $R \sim \frac{\Delta x}{kA}$

$\hookrightarrow$ as shown in BS&L ch 8, numerical corresp. as well for <u>metals</u> as both E & elec. carried <u>by</u> electrons $\Rightarrow$ ht transf. resis $\sim$ elec. resis.

Heating bills are too high, so propose to add insulation. Now looks like



still have eq^ns

$\frac{\partial^2 T}{\partial x^2} = 0$ in <u>both</u> <u>regions</u> $(q_x = cst)$

In region I : $T = T_0 + \dfrac{x - x_0}{x_1 - x_0}(T_1 - T_0)$

II : $T = T_1 + \dfrac{x - x_0}{x_1 - x_0}(T_2 - T_1)$

Determine T_1 by condition that $q_x = cst$ for both regions

$$\therefore \quad -K^{01}\frac{T_1 - T_0}{x_1 - x_0} = -K^{12}\frac{T_2 - T_1}{x_2 - x_1}$$

Thus $T_1 = T_0 + (T_2 - T_0)\left(1 - \dfrac{K^{12}}{K^{01}}\dfrac{x_1 - x_0}{x_2 - x_1}\right)$

and $\dot{Q} = (T_2 - T_0)\left(\dfrac{1}{\dfrac{x_1 - x_0}{AK^{01}} + \dfrac{x_2 - x_1}{AK^{12}}}\right)$

Equiv. to conduction in a wire : (series)

$R_{12} = \dfrac{x_2 - x_1}{AK^{12}}$

$R_{01} = \dfrac{x_1 - x_0}{AK^{01}}$

suppose wall was brick — $\therefore K^{01} = 1.5 \times 10^{-3}$ cal/sec cm°K

and insulation is $K^{12} = 10^{-4}$ cal/sec cm°K

Thus if we have 5" brick & 2" insulation, reduce cond. losses by:

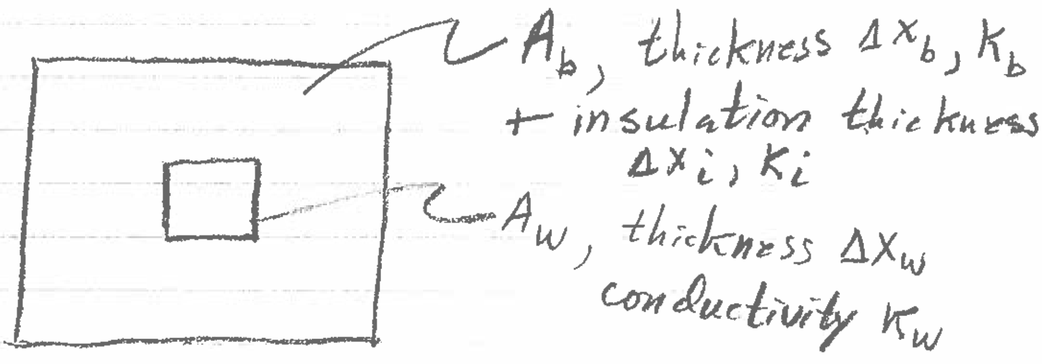
$$\frac{\dfrac{x_2 - x_1}{A\,k^{12}}}{\dfrac{x_2 - x_1}{A\,k^{12}} + \dfrac{x_1 - x_0}{A\,k^{10}}} = .14 \text{ , a factor of 7!}$$

Actual calculation more complex as should include transp. losses thru films, etc.

but shows lge effect of first bit of insulation.

End Lec. 2

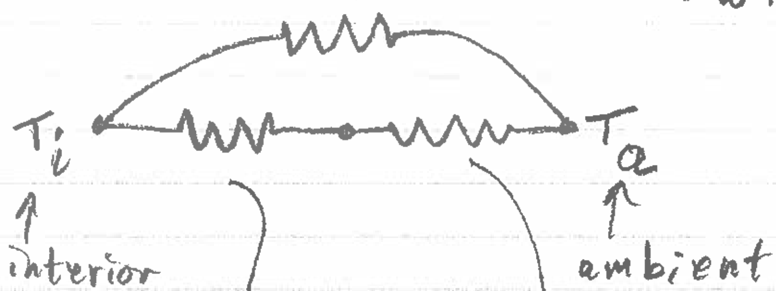
Can similarly work problems for resistances in parallel: wall w/ window



What is energy loss?

Draw electrical circuit:

$$R_w = \frac{\Delta x_w}{A_w K_w}$$

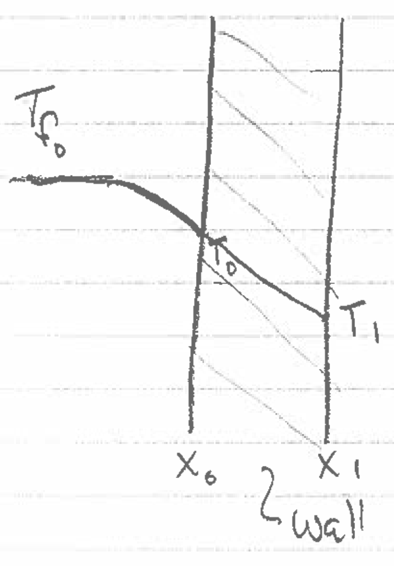


$$R_b = \frac{\Delta x_b}{A_b K_b} \qquad R_i = \frac{\Delta x_i}{A_b K_i}$$

$$\therefore\ R = \Bigg[\underbrace{\frac{1}{R_b + R_i} + \frac{1}{R_w}}_{\text{parallel resistors}}\Bigg]^{-1}$$

Thus $\dot{Q} = \dfrac{(T_i - T_a)}{R}$!

Conduction thru films: ht transf. coef.



Take case of forced convection

$\therefore$ flow indep. of T profile

As before, ht flux (now both conv. & cond.) thru any plane $x = cst.$ is same

$\therefore$ Define ht. transf. coef. h s.t.

$$q_x = h\,(T_f - T_0)$$

for transf. from fluid to surface

Still don't know what h is $\Rightarrow$ calc. is complex, will study later.

Forced conv.: h indep of $(T_f - T_0)$ $\Rightarrow$ prop. to K_f & $f^{\underline{n}}$ (flow)

Free conv: $h = f^{\underline{n}}\,(T_f - T_0)$ & sol'n is more difficult

Obtain h from correlations

Solve for total energy transf:

In solid, $T = T_0 + (T_1 - T_0)\dfrac{x - x_0}{x_1 - x_0}$

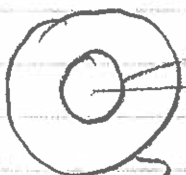
and $q_x = \dfrac{T_1 - T_0}{x_1 - x_0}\,K^{01}$

Thus $Q = (T_f - T_1)\left(\dfrac{1}{\dfrac{1}{hA} + \dfrac{\Delta x_{01}}{K^{01}A}}\right)$

$\Rightarrow$ resistance in series!

Let's look at insulation problem again, only this time in a pipe:

BS&L 9.6-1



$\leftarrow T = T_0$ at $r = r_0$

$\leftarrow T_{f_0}$ inside temp

$T = T_1$ at $r = r_1$

$T_{f_1} \sim$ surrounding temp

Look at energy transp:

S.S., so $\nabla^2 T = 0$

problem symmetric, cylindrical, so use cylindrical coord

$$\frac{d}{dr}\left(r \frac{dT}{dr}\right) = 0 \quad \text{in insulation}$$

Integrate to get:

$$T = T_0 + \frac{T_1 - T_0}{\ln(r_1/r_0)} \ln\left(\frac{r}{r_1}\right) \quad \text{in insulation}$$

$$\therefore \; Q = \frac{T_1 - T_0}{\left(\dfrac{\ln(r_1/r_0)}{2\pi L k^{01}}\right)} \Longleftrightarrow \frac{\Delta T}{R}$$

Assume some h_0 at inside wall, h_1 at outside

$\therefore$

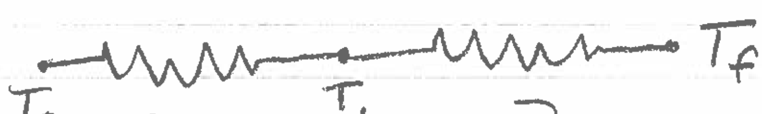
$$Q = \frac{T_{f_0} - T_{f_1}}{\frac{1}{2\pi r_0 L h_0} + \frac{\ln(r_1/r_0)}{2\pi L K^{01}} + \frac{1}{2\pi r_1 L h_1}}$$

since resistances are in series

Critical insulation thickness:

Assume resistance inside $\left(\frac{1}{2\pi r_0 L h_0}\right)$ is negligible compared to insulation $\Rightarrow$ good for, say, steam condensing on metal wall.

$\therefore$ Have situation



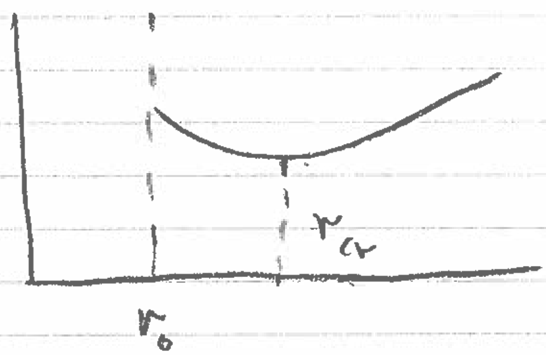
$$\underbrace{\frac{\ln(r_1/r_0)}{2\pi L K_{ins}}}_{R_{01}} \qquad \underbrace{\frac{1}{2\pi r_1 L h}}_{R_f}$$

$\therefore R = R_{01} + R_f$

Thus $R = \dfrac{\ln(r_1/r_o)}{2\pi L K_{ins}} + \dfrac{1}{2\pi r_1 L h}$

plot R vs. r_1 :

find



for certain values of K_{ins}, r_o

Adding insulation makes resistance go down!

Result of larger ht. trans f. surface on outside.

Find $r_{cr} = \dfrac{K_{ins}}{h_f}$

if $K_{ins} < r_o h_f$, R will be monotonically increasing.

Interesting effect, can obtain same effect in spheres.

Steaty S. conduction w/ ht. gen:

BS&L 9.2: conduction in a wire

Resistance dissip. gives uniform ht. gen.
(at least for DC current)

$$S_e = \frac{I^2}{K_e}$$

$I \rightarrow$ current density (amps/cm²)

$K_e \rightarrow$ electrical conductivity ($\frac{1}{\text{ohm-cm}}$)

$$\therefore \rho \hat{C}_p \frac{\partial T}{\partial t} = k \nabla^2 T + S_e$$

assumes const. properties

Assume S.S., $\therefore \frac{\partial T}{\partial t} = 0$

by symmetry, $f^{\underline{n}}$ (r) only $\therefore$

$$\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{I^2}{K_e} = 0$$

$$\therefore r\frac{dT}{dr} = -\frac{I^2}{K_e k}\frac{r^2}{2} + C_1$$

$$\text{or } \frac{dT}{dr} = \frac{-I^2}{K_e k}\frac{r}{2} + \cancel{\frac{C_1}{r}}$$

$C_1 = 0$ since $\frac{dT}{dr}$ = finite at $r = 0$

$$\therefore T = C_2 - \frac{I^2 r^2}{4 K_e k}$$

we need B.C., let $T = T_0$ at $r = R$

$$\therefore\ T - T_0 = \frac{I^2 R^2}{4 k_e k}\left(1 - \frac{r^2}{R^2}\right)$$

(similar to Poiseuille flow in tube!)

what is energy flux?

$$q_r = -k\frac{\partial T}{\partial r} = \frac{I^2}{2k_e}\, r$$

$$\therefore\ Q_R = 2\pi R L \cdot \frac{I^2 R}{2k_e} = \underbrace{\pi R^2 L}_{\text{volume}} \cdot \underbrace{S_e}_{\text{source/unit volume}}$$

Let's generalize this result

consider surface ∂D enclosing domain D

At S.S., what is net energy transfered ~~fro~~ from D ~~from~~ to surrounding solid? ~~(reverse)~~

$$\dot{Q} = \int_{\partial D} -k\, \underset{\sim}{\nabla} T \cdot \underset{\sim}{n}\, dA$$

$$= \int_D -\underset{\sim}{\nabla} \cdot (k\, \underset{\sim}{\nabla} T)\, dV = \int_D \dot{S}_{th}\, dV$$

from eq^n of energy

In above prob. we assumed const. k, k_e

How do we treat prob. if $k, k_e = f^{n}(T)$?

we have $\underset{\sim}{\nabla} \cdot \underset{\sim}{q} = S_{th} = k_e \dfrac{E^2}{L^2}$ ← voltage drop

$\underset{\sim}{q} = -k \underset{\sim}{\nabla} T \quad \therefore \quad \underset{\sim}{\nabla} \cdot (k \underset{\sim}{\nabla} T) = -S_e$

w/ B.C. $T = T_0$ at $r = R$, $\frac{\partial T}{\partial r}$ = finite at $r = 0$

Problem is non-linear, ∴ <u>very</u> difficult to solve!

Let's render dimensionless

Let $k\big|_{T=T_0} = k_0$, $\quad k_e\big|_{T=T_0} = k_{e_0}$

and let $\xi = r/R$, $\quad \Theta = \dfrac{T - T_0}{T_0}$

$$\therefore \quad -\frac{1}{\xi} \frac{\partial}{\partial \xi}\left[\frac{k}{k_0}\, \xi \,\frac{\partial \Theta}{\partial \xi}\right] = B \,\frac{k_e}{k_{e0}}$$

Where $B \equiv \dfrac{k_{e_0}}{k_0} \dfrac{R^2 E^2}{L^2 T_0}$

To obtain sol'n, we will perform regular perturbation analysis, assuming $B \ll 1$

Since we have B.C. $\Theta\big|_{\eta=1} = 0$, we recognize that to leading order,

$\Theta \sim B$

slightly different from form in book

$\therefore$ Let $\Theta = B\,\Theta_0 + B^2\Theta_1 + \ldots$

and expand K/K_0, $\frac{K_e}{K_0}$ in a Taylor series:

$$\frac{K}{K_0} = 1 - \alpha_1 \Theta - \alpha_2 \Theta^2 - \ldots$$

$$\frac{K_e}{K_{e_0}} = 1 - \beta_1 \Theta - \beta_2 \Theta^2 - \ldots$$

End 3

Now substituting in:

$$-\frac{1}{\xi}\frac{d}{d\xi}\left[\left(1 - \alpha_1\left(B\Theta_0 + B^2\Theta_1 + \ldots\right) - \alpha_2\left(B\Theta_0 + \ldots\right)^2 - \ldots\right)\cdot \xi \frac{d}{d\xi}\left(B\Theta_0 + B^2\Theta_1 + B^3\Theta_2 + \ldots\right)\right]$$

$$= B\left(1 - \beta_1\left(B\Theta_0 + B^2\Theta_1 + \ldots\right) - \beta_2\left(B\Theta_0 + \ldots\right)^2 - \ldots\right)$$

which is a mess.

Let's compare like order terms:

To $O(\beta)$:

$$-\frac{1}{\xi}\frac{d}{d\xi}\left[\xi\,\frac{d\,\cancel{\beta}\Theta_0}{d\xi}\right] = \cancel{\beta}$$

w/ B.C. $\Theta_0\Big|_{\xi=1} = 0$

$O(\beta^2)$:

$$-\frac{1}{\xi}\frac{d}{d\xi}\left[(-\alpha_1\Theta_0\,\xi\,\frac{d\Theta_0}{d\xi} + \xi\,\frac{d\Theta_1}{d\xi}\right]$$

$$= -\beta_1\Theta_0$$

or $$-\frac{1}{\xi}\frac{d}{d\xi}\left[\xi\,\frac{d\Theta_1}{d\xi}\right] = \underbrace{-\beta_1\Theta_0 + \frac{1}{\xi}\frac{d}{d\xi}\left[\alpha_1\Theta_0\,\xi\,\frac{d\Theta}{d\xi}\right.}_{\text{only in terms of }\Theta_0\text{, so known}}$$

linear in Θ_1

w/ B.C. $\Theta_1\Big|_{\xi=1} = 0$

Now for $O(\beta^3)$:

$$-\frac{1}{\xi}\frac{d}{d\xi}\left[\xi\,\frac{d\Theta_2}{d\xi}\right] = -\beta_1\Theta_1 - \beta_2\Theta_0^2$$

(cont. next pg)

$$-\frac{1}{\zeta}\frac{d}{d\zeta}\left[\alpha_1\left(\Theta_0\,\zeta\frac{d\Theta_1}{d\zeta}+\Theta_1\,\zeta\frac{d\Theta_0}{d\zeta}\right)+\alpha_2\,\Theta_0^2\,\zeta\frac{d\Theta_0}{d\zeta}\right]$$

w/ B.C. $\Theta_2'\Big|_{\zeta=1}=0$ $\Rightarrow$ increasingly complex

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Let's obtain first 2 terms:

$$-\frac{1}{\zeta}\frac{d}{d\zeta}\left[\zeta\frac{d\Theta_0}{d\zeta}\right]=1\,,\quad \Theta_0\Big|_{\zeta=1}=0$$

$\therefore\ \Theta_0=\frac{1}{4}(1-\zeta^2)$ from previous problem

Now for 2nd term:

$$-\frac{1}{\zeta}\frac{d}{d\zeta}\left[\zeta\frac{d\Theta_1}{d\zeta}\right]=-\beta_1\Theta_0-\frac{1}{\zeta}\frac{d}{d\zeta}\left[\alpha_1\Theta_0\,\zeta\frac{d\Theta_0}{d\zeta}\right]$$

$$=-\frac{\beta_1}{4}(1-\zeta^2)-\frac{1}{\zeta}\frac{d}{d\zeta}\left[\frac{\alpha_1}{4}(1-\zeta^2)\,\zeta\frac{d}{d\zeta}\left(\frac{1}{4}(1-\zeta^2)\right)\right]$$

Integrating once:

$$\xi \frac{d\Theta_1}{d\xi} = C_1 + \frac{\beta_1}{4}\left(\frac{1}{2}\xi^2 - \frac{1}{4}\xi^4\right) + \left[\frac{\alpha_1}{4}(1-\xi^2)\left(-\frac{\xi^2}{2}\right)\right]$$

or $$\frac{d\Theta_1}{d\xi} = \cancel{\frac{C_1}{\xi}} + \frac{\beta_1}{4}\xi\left(\frac{1}{2} - \frac{1}{4}\xi^2\right) - \frac{\alpha_1}{8}\xi(1-\xi^2)$$

$C_1 \to 0$ as must be finite for $\xi = 0$

Thus:

$$\Theta_1 = C_2 + \frac{\beta_1}{4}\left(\frac{1}{4}\xi^2 - \frac{1}{16}\xi^4\right) - \frac{\alpha_1}{8}\left(\frac{1}{2}\xi^2 - \frac{1}{4}\xi^4\right)$$

Now $\Theta_1 = 0$ at $\xi = 1$

$$\therefore \quad 0 = C_2 + \frac{\beta_1}{16}\frac{3}{4} - \frac{\alpha_1}{32}$$

Thus $$C_2 = -\frac{\beta_1 3}{64} + \frac{\alpha_1}{32}$$

And, after rearranging, we obtain:

$$\Theta_1 = \frac{1}{4}(1-\zeta^2)\left(-\frac{\beta_1}{16}(3-\zeta^2) + \frac{\alpha_1}{8}(1-\zeta^2)\right)$$

or

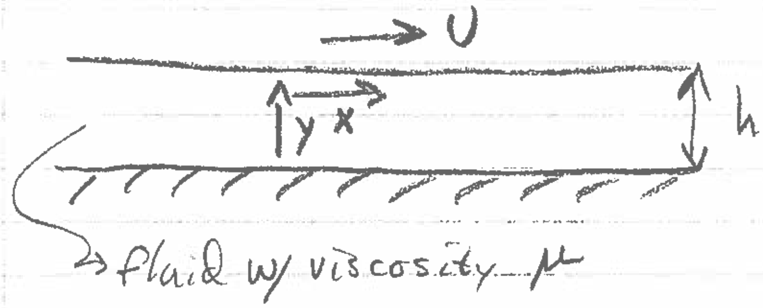
$$\Theta = \frac{B}{4}(1-\zeta^2)\left[1 + B\left[-\frac{\beta_1}{16}(3-\zeta^2) + \frac{\alpha_1}{8}(1-\zeta^2)\right] + O(B^2)\right]$$

This is a <u>regular</u> perturbation expansion $\Rightarrow$

does not always work this well. Often, can't satisfy all B.C.'s at higher order $\Rightarrow$ results in singular perturbation.

Often approach this type of problem using matched assymptotic expansion $\Rightarrow$ will discuss later.

Another regular perturbation problem: viscous heating in bearings



what is effect of viscous heating?

we have the equation of momentum:

$$\rho \frac{D\underset{\sim}{u}}{Dt} = -\underset{\sim}{\nabla} p + \underset{\sim}{\nabla} \cdot \underset{\approx}{\tau} + \rho \underset{\sim}{g}$$

Assume S.S., $\underset{\sim}{u} \equiv u\, \underset{\sim}{\hat{e}}_x$, no variation in x, z Dir.

$$\therefore \quad \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0 \quad \text{w/} \quad u\Big|_{y=0} = 0, \quad u\Big|_{y=h} = 1$$

And eq'n of energy:

$$\rho \hat{C}_p \frac{DT}{Dt} = \underset{\sim}{\nabla} \cdot \left(K \underset{\sim}{\nabla} T\right) + \mu \Phi_v$$

(assume newtonian, incompressible)

$$\therefore \Phi_v = \left(\frac{\partial u}{\partial y}\right)^2$$

Thus:

$$0 = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \mu \left(\frac{\partial u}{\partial y}\right)^2$$

w/ $T = T_0$ at $y=0$, $y=h$

Now if $\mu, k \neq f^n(T)$, problem is <u>uncoupled</u>:

Solve momentum transp. eq'n, then E eq'n.

i.e., $u = U \frac{y}{h}$, $T = T_0 + \frac{1}{2}\frac{\mu U^2}{k}\left(\frac{y}{h} - \frac{y^2}{h^2}\right)$

but we are interested in effect of non-constant properties!

Thus we obtain:

$$\frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) = -\mu \left(\frac{\partial u}{\partial y}\right)^2 \qquad T\Big|_{y=0,\, y=h} = T_0$$

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0 \qquad u\Big|_{y=0} = 0 \quad u\Big|_{y=h} = U$$

Let's render problem dimensionless

let $\mu\Big|_{T_0} = \mu_0$, $k\Big|_{T_0} = k_0$, $y^* = y/h$, $T^* = \dfrac{T - T_0}{\underbrace{\frac{\mu_0 U^2}{k_0}}_{\text{char } \Delta T}}$

and $u^* = u/U$

Thus:

$$\frac{d}{dy^*}\left(\frac{K}{K_0}\frac{dT^*}{dy^*}\right) = -\frac{\mu}{\mu_0}\left(\frac{du^*}{dy^*}\right)^2 ; \quad T^*\Big|_{y^*=0,1} = 0$$

and

$$\frac{d}{dy^*}\left(\frac{\mu}{\mu_0}\frac{du^*}{dy^*}\right) = 0 \quad ; \; u^*\Big|_{y^*=0} = 0, \; u^*\Big|_{y^*=1} = 1$$

End 4 we have a coupled non-linear problem!

Let's solve using a regular perturbation approach.

Let $u^* = u^0 + N u^1 + N^2 u^2 + \ldots$

where $N = \frac{\mu_0 U^2}{K_0 T_0} \ll 1$

T_0 ↳ Dimensionless char. ΔT

and $T^* = T^0 + N T^1 + N^2 T^2 + \ldots$

we also need a Taylor series expansion for $\frac{\mu}{\mu_0}, \frac{K}{K_0}$

$$\therefore \frac{\mu}{\mu_0} = 1 + \frac{\mu'}{\mu_0}(T - T_0) + \ldots$$

$$= 1 + N T_0 \frac{\mu'}{\mu_0} T^* + O(N^2)$$

and

$$\frac{K}{K_0} = 1 + N T_0 \frac{K'}{K_0} T^* + O(N^2)$$

$$= 1 + N\, T_0 \frac{K'}{K_0} T^0 + O(N^2)$$

End 4/88

Now we have the problem:

$$O(0): \quad \frac{\partial}{\partial y^*}\left(\frac{\partial T^0}{\partial y^*}\right) = -\left(\frac{\partial u^0}{\partial y^*}\right)^2; \quad T^0\Big|_{y^*=0,1} = 0$$

$$\text{and} \quad \frac{\partial}{\partial y^*}\left(\frac{\partial u^0}{\partial y^*}\right) = 0; \quad u^0\Big|_{y=0} = 0, \quad u^0\Big|_{y^*=1} = 1$$

which has the solution:

$$u^0 = y^*, \quad T^* = \tfrac{1}{2}\left(y^* - y^{*2}\right)$$

and to next order:

$$O(1): \quad \frac{\partial}{\partial y^*}\left(T_0 \frac{K'}{K_0} T^0 \frac{\partial T^0}{\partial y^*} + \frac{\partial T'}{\partial y^*}\right) = -T_0 \frac{\mu'}{\mu_0} T^0 - 2\left(\frac{\partial u^0}{\partial y^*}\right)\left(\frac{\partial u'}{\partial y^*}\right)$$

w/ B.C.'s: $T'\Big|_{0,1} = 0$

And:

$$\frac{\partial}{\partial y^*}\left(T_0 \frac{\mu'}{\mu_0} T^0 \frac{\partial u^0}{\partial y^*} + \frac{\partial u'}{\partial y^*}\right) = 0$$

w/ B.C.'s $u'\Big|_{y^*=0,1} = \underline{\underline{0}}$ ← note: zero at $y^*=1$!

Problem is now uncoupled: may solve momentum first, then solve for temp.

Simple matter to integrate & obtain sol'n.

Solving for u':

$$T_0 \frac{\mu'}{\mu_0} \frac{1}{2}(y^* - y^{*2}) + \frac{\partial u'}{\partial y^*} = C_1$$

$$\therefore \; u' = C_1 y^* + C_2 - T_0 \frac{\mu'}{\mu_0} \frac{1}{2}\left(\frac{1}{2} y^{*2} - \frac{1}{3} y^{*3}\right)$$

$$u'\Big|_{y^*=0} = 0 \quad \therefore C_2 = 0$$

$$u'\Big|_{y^*=1} = 0 = C_1 - T_0 \frac{\mu'}{\mu_0} \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right)$$

Thus
$$u' = T_0 \frac{\mu'}{\mu_0} \frac{1}{2}\left(\frac{1}{6} y^* - \frac{1}{2} y^{*2} + \frac{1}{3} y^{*3}\right)$$
$$= T_0 \frac{\mu'}{\mu_0} \frac{1}{12} y^* (1 - y^*)(2 - y^*)$$

The observed shear stress is thus:

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} = \mu_0 \left(1 + T_0 N \frac{\mu'}{\mu_0} T^0 + O(N^2)\right) \cdot \left(\frac{\partial u^0}{\partial y} + N \frac{\partial u'}{\partial y} + O(N^2)\right)$$

$$= \mu_0\left[\frac{\partial u^0}{\partial y} + N\left(T_0 \frac{\mu'}{\mu_0} T^0 \frac{\partial u^0}{\partial y} + \frac{\partial u'}{\partial y}\right) + O(N^2)\right]$$

$$= \frac{\mu_0 U}{h}\left[1 + N\left(T_0\frac{\mu'}{\mu_0}\frac{1}{2}(\cancel{y^*} - y^{*2}) + T_0 \frac{\mu'}{\mu_0}\frac{1}{12}(1 - \cancel{6y^*} + 6y^{*2})\right)\right]$$

$$= \frac{\mu_0 U}{h}\left[1 + \frac{1}{12} N \left.\frac{\partial \ln\mu}{\partial \ln T}\right|_{T_0} + O(N^2)\right]$$

Note: indep. of y^* as it should be

Plug in #'s : (My Rheo)

1" shaft at 3600 rpm $\Rightarrow$ $U = 480$ cm/sec

heavy machine oil: $\mu_0 = 3.3\,p$, $\left.\frac{\partial \ln\mu}{\partial \ln T}\right|_{25°C} = -22$

$k_0 = 1.4 \times 10^4 \; \frac{erg}{sec\, cm\, °K}$ $\quad T_0 = 300°K$

Thus $N = .18$

and $T_{max} = T_0 + \frac{N}{8} T_0 + O(N^2)$

$$= T_0 + 6.8\,^\circ K + O(N^2)$$

with a torque reduction of

$$\frac{1}{12} N \left.\frac{d \ln \mu}{d \ln T}\right|_{T_0} = \underline{-33\%} \,!$$

so small ΔT's can have lge. effects!

Now look at Convective energy transp.

Prob. 10.k, BS&L : Transpirational Cooling

Increased efficiency LOX container :