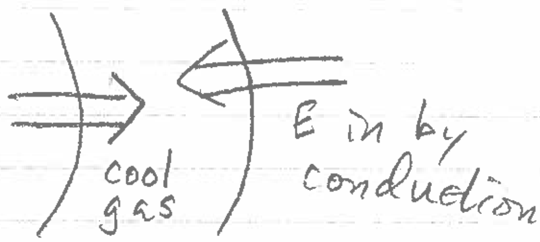
As E seeps in, O_2 boils. Normally just vented to atm, what is decrease in LOX loss rate if O_2 forced thru insulation?

To solve, assume thermal eq. between O_2 & insulation in contact (temp same in two phases)



pseudo-SS., so $\frac{\partial}{\partial t} = 0$

spherical symmetry

If we have porosity ε (ratio of pore volume to total volume), then C.E:

$$\varepsilon \frac{\partial \rho}{\partial t} = -(\underset{\sim}{\nabla} \cdot \rho \underset{\sim}{u}^{\circ})$$

where $\underset{\sim}{u}^{\circ}$ is <u>superficial</u> velocity $\equiv$ volume rate of flow thru unit area X-section of solid $\Rightarrow$ ave over area lge w.r.t. pore size.

Reduces to $\frac{\partial}{\partial r}\left(\rho r^2 u_r^o\right) = 0$

$\therefore \rho r^2 u_r^o = cst = \frac{\omega_r}{4\pi}$ → mass flow rate (determined by boiling)

Eq'n of motion in porous media given by Darcy's Law:

$$\underset{\sim}{u}^o = -\frac{K}{\mu}\left(\underset{\sim}{\nabla} p - \rho \underset{\sim}{g}\right)$$

where K is permeability of medium

To solve for pressure drop, need ρ, $\mu(T)$, thus require sol'n to energy eq'n ⇒ coupled & non-linear

To simplify, assume some ave ht. capacity for O_2 & thermal cond. for insulation, thus:

$$\rho \hat{C}_p u_r^o \frac{dT}{dr} = k \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right)$$

or

$$\hat{C}_p \underbrace{\left(\rho u_r^o r^2\right)}_{\frac{\omega_r}{4\pi} = \underline{cst}} \frac{dT}{dr} = k \frac{d}{dr}\left(r^2 \frac{dT}{dr}\right)$$

or $\frac{\partial T}{\partial r} = \left(\frac{4\pi k}{\omega_r \hat{C}_p}\right) \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)$

Render dimensionless:

$T^* = \frac{T - T_0}{T_1 - T_0}$, $r^* = r/R_1$

Thus $N \frac{\partial T^*}{\partial r^*} = \frac{\partial}{\partial r}\left(r^{*2} \frac{\partial T^*}{\partial r^*}\right)$

where $N = \frac{\omega_r \hat{C}_p}{4\pi K R_1} \equiv \frac{\text{convection}}{\text{conduction}}$

End lec 5

Integrating above eq'n:

$T^* = C_1 + C_2 e^{-N/r^*}$

w/ B.C.'s at $r^* = \frac{R_0}{R_1}$, $T^* = 0$

$r^* = 1$, $T^* = 1$

Thus $0 = C_1 + C_2 e^{-N\frac{R_1}{R_0}}$

$1 = C_1 + C_2 e^{-N}$

hence:

$C_2 = \frac{1}{e^{-N} - e^{-N R_1/R_0}}$, $C_1 = \frac{-e^{-N R_1/R_0}}{e^{-N} - e^{-N R_1/R_0}}$

or

$$\frac{T - T_0}{T_1 - T_0} = \frac{e^{-N\frac{R_1}{r}} - e^{-N R_1/R_0}}{e^{-N} - e^{-N R_1/R_0}}$$

The energy flux into the tank is given by:

$$Q = -4\pi r^2 q_r \Big|_{r=R_0} = 4\pi K r^2 \frac{\partial T}{\partial r}\Big|_{r=R_0}$$

$$= 4\pi K R_1 N \cdot \frac{T_1 - T_0}{e^{N\left(R_1/R_0 - 1\right)} - 1}$$

we wish to compare ht. gain w/ transp. to that without,

For pure conduction, simply take limit as $N \to 0$ (i.e. $\omega_r = 0$)

$$\therefore\ Q_{cond} = \lim_{N\to 0} (T_1 - T_0) \frac{4\pi K R_1 \cancel{N}}{\cancel{1} + \cancel{N}\left(R_1/R_0 - 1\right) - \cancel{1}}$$

$$= (T_1 - T_0)\frac{4\pi K R_1}{R_1/R_0 - 1}$$

Thus if l is latent heat of vaporization:

$$(w_r)_{cond} = \frac{Q_{cond}}{l} = \frac{4\pi K R_1}{l(R_1/R_0 - 1)}(T_1 - T_0)$$

similarly, for transp,

$$w_r = \frac{4\pi K R_1 N}{l} \frac{T_1 - T_0}{e^{N(R_1/R_0 - 1)} - 1}$$

$$= \frac{w_r \hat{C}_p}{l} \frac{T_1 - T_0}{e^{N(R_1/R_0 - 1)} - 1}$$

$$\therefore \quad e^{N(R_1/R_0 - 1)} = 1 + \frac{\hat{C}_p(T_1 - T_0)}{l}$$

or

$$w_r = \frac{4\pi K R_1}{\hat{C}_p(R_1/R_0 - 1)} \ln\left[1 + \frac{\hat{C}_p(T_1 - T_0)}{l}\right]$$

thus

$$\frac{w_r}{(w_r)_{cond}} = \left(\frac{\hat{C}_p\cancel{(T_1 - T_0)}}{l\cancel{(T_1 - T_0)}}\right)^{-1} \ln\left[1 + \frac{\hat{C}_p(T_1 - T_0)}{l}\right]$$

Note: indep. of R_1, R_0 & K! (although total mass loss <u>is</u> f^n of these variables

If $T_0 = -297°F$, $T_1 = 30°F$

$\hat{C}_p = .22 \ \text{Btu}/\text{lb}\,°F$, $\ell = 91.7 \ \text{Btu}/\text{lb}$,

find

$$\frac{w_r}{(w_r)_{cond}} = .74 \quad \therefore \text{ 26\% less } O_2 \text{ lost for free!}$$

Usual to define a transp. efficiency:

$$\zeta \equiv \frac{\left| Q\Big|_{R_0}^{cond} \right| - \left| Q\Big|_{R_0}^{tr.} \right|}{\left| Q\Big|_{R_0}^{cond} \right|}$$

$$= 1 - \frac{\phi}{e^{\phi} - 1} \sim \phi/2 \text{ for small } \phi$$

where $\phi = \dfrac{w_r \hat{C}_p}{4\pi k}\left(\dfrac{R_1 - R_0}{R_1 R_0}\right)$

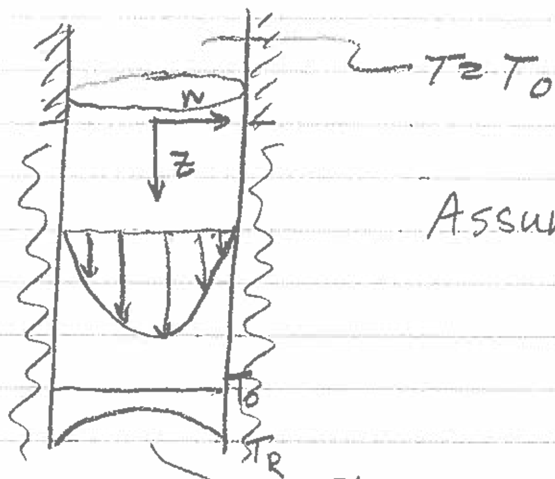
for boiling O_2, w_r is fixed, although in general w_r could be any value

Now look at more complex forced convection problem:

Forced Convection in a Pipe

(BS&L 9.8)

Nusselt - Graetz problem
(const. wall ht flux)



Assume S.S., laminar flow
(fully developed)

T low in center,
high at walls

Assume const. properties, so may decouple eq'n of motion from energy

$$0 = -\frac{\partial p}{\partial z} - \rho g + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) \Big\} \text{ z-mom.}$$

Define $P = p + \rho g z$

$$\therefore \frac{\mu}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right) = \frac{dP}{dz} = cst$$

just Poiseuille flow:

$$u_z = u_z^{(max)}\left(1 - \frac{r^2}{R^2}\right); \quad u_z^{(max)} = \frac{P_0 - P_L}{L}\frac{R^2}{4\mu}$$

Now we have eq'n of Energy

$$\underbrace{\rho \hat{C}_p u_z \frac{\partial T}{\partial z}}_{\text{axial convection}} = \underbrace{\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)}_{\text{radial conduction}} + \underbrace{k\frac{\partial^2 T}{\partial z^2}}_{\text{axial cond.}} + \underbrace{\mu\left(\frac{\partial u_z}{\partial r}\right)^2}_{\text{viscous dissip. (small)}}$$

$$+ \text{ B.C. } \quad T\Big|_{z=-\infty} = T_0, \quad -k\frac{\partial T}{\partial r}\Big|_{r=R,\, z>0} = q, \quad \frac{\partial T}{\partial r}\Big|_{r=R,\, z<0} = 0$$

Let's render problem dimensionless:

Assume some char. length z-dir L (to be det.) $\therefore$ $z^* = z/L$, $r^* = r/R$

get from eq'n

$$T^* = \frac{T - T_0}{q\, R/k}$$

Thus:

$$u_z^* \frac{\partial T^*}{\partial z^*} = \left(\frac{L\, k}{\rho \hat{C}_p u_z^{max} R^2}\right)\left[\frac{\partial}{\partial r^*}\left(r^* \frac{\partial T^*}{\partial r^*}\right) + \frac{R^2}{L^2}\frac{\partial^2 T^*}{\partial z^{*2}}\right]$$

w/ B.C.'s

$T^* =$ finite at $r^* = 0$

$\frac{\partial T^*}{\partial r^*} = -1$ at $r^* = 1$, $z^* > 0$

$\frac{\partial T^*}{\partial r^*} = 0$ at $r^* = 1$, $z^* < 0$

and $T^* = 0$ as $z^* \to -\infty$

What is appropriate choice for L?

wish to balance convection & conduction

convection in z-dir, conduction in R-dir.

so choose

$$\frac{L\, K}{\rho\, \hat{C}_p\, u_z^{max} R^2} = 1$$

and we obtain:

$$u_z^* \frac{\partial T^*}{\partial z^*} = \frac{\partial}{\partial r^*}\left(r^* \frac{\partial T^*}{\partial r^*}\right) + \left[\frac{K}{\rho\, \hat{C}_p\, u_z^{max} R}\right]^2 \frac{\partial^2 T^*}{\partial z^{*2}}$$

What is term in brackets? $\Rightarrow$ Rel. imp. of z-dir conduction to z-dir convection.

Now we may rewrite the term in brackets:

$$\left[\frac{k}{\rho \hat{C}_p u_z^{max} R}\right]^2 = \left(\left(\frac{u_z^{max} R}{\nu}\right)\left(\frac{\nu}{\alpha}\right)\right)^{-2}$$

$$= (Re\,Pr)^{-2}$$

$$Re = \frac{\text{convection}}{\text{viscous diffusivity}} \qquad Pr = \frac{\text{viscous dif.}}{\text{thermal dif.}}$$

$$\therefore\ Re\,Pr = \frac{\text{convection}}{\text{thermal dif}}$$ ⇒ shows up in convection probs.

In most situations, $Re\,Pr$ is lge, ∴ ignore conduction in z-dir. rel. to convection

Thus,

$$u_z^* \frac{\partial T^*}{\partial z^*} = \frac{1}{r^*}\left(\frac{\partial}{\partial r^*}\left(r^* \frac{\partial T^*}{\partial r^*}\right)\right)$$

B.C.'s: T^* finite at $r^*=0$, $T^*=0$ $z^* \to -\infty$

$$\left.\frac{\partial T^*}{\partial r^*}\right|_{r^*=1} = \begin{cases} 0 & z^*<0 \\ -1 & z^*>0 \end{cases}$$

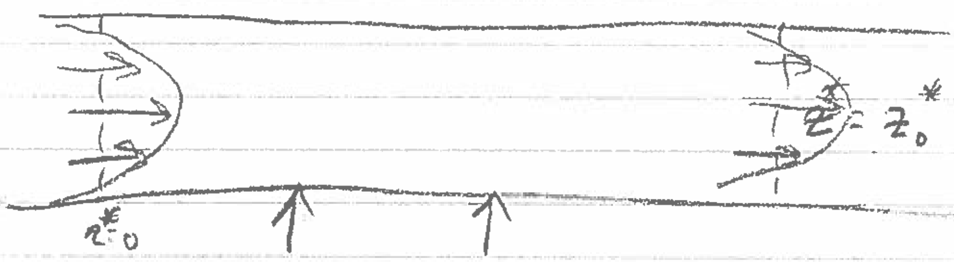
Rather nasty for small z^*, look at behavior for $z^* \gg 1$ first

we expect T^* to rise linearly in z^*

Assume $T^* = C_0 z^* + \psi(r^*)$, $z^* > 0$

where C_0 will be determined from energy balance over entire pipe.

This expression <u>cannot</u> satisfy cond. at $z = 0$, so use E balance instead.



thus $2\pi \int_0^R \rho \hat{C}_p (T - T_0) v_z \, r \, dr = -2\pi R z \varepsilon$

End lec. 6

or $-z^* = \int_0^1 T^* (1 - r^{*2}) r^* dr^*$

Thus, we obtain:

$$\frac{1}{r^*}\frac{d}{dr^*}\left(r^* \frac{d\psi}{dr^*}\right) = \overbrace{C_0}^{\frac{\partial T^*}{\partial z^*}} (1 - r^{*2})$$

Integration yields:

$$T^* = C_0 z^* + \frac{C_0 r^{*2}}{4}\left(1 - \frac{r^{*2}}{4}\right) + C_1 \ln r^* + C_2$$

for $z^* >> 1$

Since $T^* =$ finite at $r^* = 0$, $C_1 = 0$

Also, $\frac{\partial T^*}{\partial r^*} = -1$ at $r^* = 1$ $\therefore$ $C_0 = -4$

and the energy balance gives $C_2 = \frac{7}{24}$
$\hookrightarrow$ equiv. to flow avg. T at $z=0$ $=0$

$\therefore$ $T^* = -4z^* - r^{*2} + \frac{1}{4} r^{*4} + \frac{7}{24}$, $z^* >> 1$

Now how do we solve for <u>all</u> z?

Let's take sol'n derived above as T_∞^*

and let $T^* = T_d^* + T_\infty^*$
T_d^* $\hookrightarrow$ decaying; T_∞^* $\hookrightarrow$ asymptotic for large z^*

Thus, have sol'n

$$(1 - r^{*2}) \frac{\partial T_d^*}{\partial z^*} = \frac{1}{r^*} \frac{\partial}{\partial r^*}\left(r^* \frac{\partial T_d^*}{\partial r^*}\right)$$

w/B.C.'s : $\left.\frac{\partial T_d^*}{\partial r^*}\right|_{r^*=1} = 0$, $\left.T_d^*\right|_{z^*=0} = -\left.T_\infty^*\right|_{z^*=0}$

and $T_d^* \to 0$ as $z^* \to \infty$

We may solve this by separation of variables

let
$$T_d^* = F(r^*)\, G(z^*)$$

$$\therefore\ (1-r^{*2}) F\, G' = \frac{G}{r^*} \frac{d}{dr^*}(r^* F')$$

$$\text{or}\quad \frac{G'}{G} = \frac{1}{F}\, \frac{\frac{d}{dr^*}(r^* F')}{r^*(1-r^{*2})} = -c^2 \ (\text{cst})$$

eigenvalue problem

$$\therefore\ G(z^*) = e^{-c^2 z^*} \qquad (\text{decays at } \infty)$$

and
$$(r^* F')' + c^2 r^* (1-r^{*2}) F = 0$$

w/ B.C.'s $F'(0) = 0$, $r^* F'\big|_{r^*=1} = 0$

Problem for $F(r^*)$ is a Sturm-Liouville bdy-value problem ⇒ Review Boyce & DiPrima

For a problem of form:
$$[p(x)\, y']' - q(x)\, y + \lambda\, w(x)\, y = 0$$

on interval $0 < x < 1$

w/ B.C.'s: (homogeneous)

$$a_1 y(0) + a_2 y'(0) = 0$$

$$b_1 y(1) + b_2 y'(1) = 0$$

and p, p', q & w are continuous $f^{\underline{n}}(x)$ on interval $0 \le x \le 1$ and that $p(x) > 0$ and $w(x) > 0$ on $[0,1]$

<u>Then</u>:

1) all eigenvalues λ & corresp. eigen$f^{\underline{n}}$s are real

2) All eigenvalues are simple: one-to-one corresp. w/ eigen$f^{\underline{n}}$s, may be ordered w/ $\lambda_n \to \infty$ as $n \to \infty$

3) Orthogonality;

$$\int_0^1 \phi_i(x)\,\phi_j(x)\,w(x)\,dx = 0 \quad i \neq j$$

4) set of eigen$f^{\underline{n}}$s are <u>complete</u>

Not quite: singular

Thus we are guaranteed sol'n to D.E. $F_n(r^*)$ exist, w/ eigenvalues C_n

s.t. $\left(r^* F_n'\right)' + C_n^2 r^*(1-r^{*2}) F_n = 0$

and wt. f^n $= w(r^*) \equiv r^*(1-r^{*2})$

Thus $T_d(r^*, z^*) = \sum_{n=1}^{\infty} B_n e^{-C_n^2 z^*} F_n(r^*)$

where, by orthog:

$$B_n = -\frac{\int_0^1 T_\infty^*(r^*, 0) F_n(r^*)\, r^*(1-r^{*2})\, dr^*}{\int_0^1 F_n^2(r^*)\, r^*(1-r^{*2})\, dr^*}$$

Sol'n for F_n very complex: for F_n up to $n=7$ see Spiegel, Sparrow & Hallman Appl. Sci. Res. A 7, 386, 1958

Now we have T distrib: what can we do with this?

We can define (at least) two mean temp.

What is avg temp across X-section?

$$\langle T \rangle = \frac{\int_0^R T \, 2\pi r \, dr}{\pi R^2}$$

Not too useful. Better avg is bulk temp (flow avg, cup mix ⇒ what get if we emptied pipe into a cup)

$$T_b = \frac{\langle u_z T \rangle}{\langle u_z \rangle} = \frac{\int_0^R T u_z \, 2\pi r \, dr}{\int_0^R u_z \, 2\pi r \, dr}$$

For Poiseuille flow, $u_z = u_z^{max}(1 - r^{*2})$

Thus $$T_b^* = \frac{T_b - T_0}{\varepsilon R/k} = 4 \int_0^1 T^*(1 - r^{*2}) r^* \, dr^*$$

Now T^* satisfies D.E.:

$$r^*(1 - r^{*2}) \frac{\partial T^*}{\partial z^*} = \frac{\partial}{\partial r^*}\left(r^* \frac{\partial T^*}{\partial r^*} \right)$$

Thus

$$\frac{\partial T_b^*}{\partial z^*} = 4\int_0^1 \frac{\partial T^*}{\partial z^*} r^*(1-r^{*2})\,dr^*$$

$$= 4\int_0^1 \frac{\partial}{\partial r^*}\left(r^* \frac{\partial T^*}{\partial r^*}\right) dr^*$$

$$= 4\; r^* \frac{\partial T^*}{\partial r^*}\bigg|_0^1 = -4 \quad \text{from B.C.}$$

since $T_b^*\big|_{z^*=0} = 0$ we obtain:

$$T_b^* = -4z^*$$

which makes physical sense ⇒ can be (and indir. was) obtained from E balance

We may calculate the wall temp:

$$T^*\big|_{r=1} = \left(T_\infty^* + T_d^*\right)\Big|_{r^*=1} = -\frac{11}{24} - 4z^* + \sum_{n=1}^{\infty} B_n e^{-c_n^2 z^*} F_n(1)$$

we are now in a position to calculate the ht. transf. coeff. h & Nusselt # (dim. ht. transf.)

Consider tube of length L,

$$\underset{\text{into fluid}}{Q} = \int_0^L \int_0^{2\pi} \left(k \frac{\partial T}{\partial r}\right)\bigg|_{r=R} R\,d\theta\,dz \equiv \underbrace{h A \Delta T}_{\text{definition!}}$$

Three common ways to define integral ht. transf. coeff.

$$Q = h_1 (2\pi RL)(T_w - T_b)\Big|_{z=z_1} \quad \text{(inlet based)}$$

$$Q = h_a (2\pi RL)\frac{1}{2}\left((T_w - T_b)\Big|_{z=z_1} + (T_w - T_b)\Big|_{z=z_2}\right)$$

average

$$Q = h_{ln} (2\pi RL)\left[\frac{(T_w - T_b)\big|_{z_1} - (T_w - T_b)\big|_{z_2}}{\ln\left[(T_w - T_b)\big|_{z_1} / (T_w - T_b)\big|_{z_2}\right]}\right]$$

log-mean ⇒ most commonly used in industry as not a strong f^n (L/D) of pipe

+ diff. ht transf. coef.

$$dQ = h_{loc} (2\pi R\, dz)(T_w - T_b)$$

which we will use

but only as $u \equiv f^n(Re)$

Note: $h \doteq f^n$ (flow (i.e. Re), B.C.'s, L/D)

End lec. 7

We can define a Nusselt #:

$$Nu = \frac{h(2R)}{k} \equiv \text{deviation of ht. transf. from pure conduction,}$$

i.e.: $Nu = \dfrac{\text{Total ht. transf.}}{\text{conductive ht. transf.}}$

For ht. pipe:

$$Nu_{loc} = \frac{h_{loc}(2R)}{K} = \frac{-K \left.\frac{\partial T}{\partial r}\right|_{r=R}}{T_W - T_b} \cdot \frac{2R}{K}$$

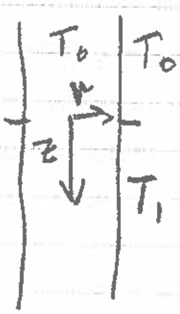
$$= \frac{2}{T_W^* - T_b^*} = \frac{2}{\frac{11}{24} + \sum_{n=1}^{\infty} B_n e^{-C_n^2 z^*} F(1)}$$

Far down the pipe, $e^{-C_n^2 z^*} \to 0$

$$\therefore \; Nu_{loc} = \frac{48}{11} = \text{cst. for fully dev. flow}$$

We may also solve analogous problem of step change in temp. at wall:

skip



Now define $u_z^* = u_z / u_z^{max}$, $r^* = r/R$, $z^* = \frac{z k}{\rho C_p R^2 u_z^{max}}$ as before & take:

$$T^* = \frac{T_1 - T}{T_1 - T_0}$$ $\Leftarrow$ Def^n chosen so that $T^* = 0$ at walls and at $z^* = \infty$

Again:

$$u_z^* \frac{\partial T^*}{\partial z^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial T^*}{\partial r^*}\right) + \left(\frac{1}{Re\,Pr}\right)^2 \frac{\partial^2 T^*}{\partial z^{*2}}$$

where $Re = \frac{u_z^{max} R}{\nu}$, $Pr = \frac{\nu}{\alpha}$

we may neglect z cond. <u>provided</u> $Re Pr >> 1$ <u>and</u> scaling for z is correct (i.e., $\frac{\partial^2 T^*}{\partial z^{*2}} = O(1)$

where is this likely to break down?

At T jump ($z=0$) z-cond. <u>must</u> play a role i.e., $z^* << 1$ $\therefore$ scaling will be incorrect in this region (likewise in htd wall prob)

Far down tube ($z^* \geq O(1)$, $Re Pr >> 1$) z-cond. won't matter & we have:

$$u_z^* \frac{\partial T^*}{\partial z^*} = \frac{1}{r^*} \frac{\partial}{\partial r^*}\left(r^* \frac{\partial T^*}{\partial r^*}\right)$$

w/ B.C.'s: $T^*\Big|_{r^*=1} = 1$, $z^* < 0$; $T^*\Big|_{\substack{r^*=1,\\ z^*>0}} = T^*\Big|_{z^* \to \infty} = 0$

and neglecting axial cond:

$$T^*\Big|_{z^*=0} = 1 \quad w/\ T^* \text{ finite } \left(\frac{\partial T^*}{\partial r^*} = 0\right) \text{ at } r^* = 0$$

As before, $T^* = F(r^*)\, G(z^*)$

Thus $(1 - r^{*2}) F G' = G (r^* F')'$

or $\frac{G'}{G} = \frac{1}{r^*(1 - r^{*2})} \frac{(r^* F')'}{F} = -c^2$

w/ $G = e^{-c^2 z^*}$

and $(r^* F')' + c^2 r^* (1 - r^{*2}) F = 0,$

$F\big|_{r^*=1} = 0, \ F'\big|_{r^*=0} = 0$

singular Sturm Liouville prob. again, just dif. B.C.

No closed form sol'n, so try series sol'n of form:

$$F = \sum_{n=0}^{\infty} A_n r^{*n}$$ & plug into ODE:

skip, use num. sol'n!

$$\sum_{n=1}^{\infty} A_n n^2 r^{*n-1} + c^2 r^* (1 - r^{*2}) \sum_{n=0}^{\infty} A_n r^{*n}$$

comparing like powers of r^* we obtain:

$$F = A_0 \left\{ 1 - \frac{c^2}{4} r^{*2} + \frac{1}{16}\left(c^2 + \frac{c^4}{4}\right) r^{*4} - \frac{1}{36 \cdot 16}\left(5c^2 + \frac{c^6}{4}\right) r^{*6} + \dots \right\}$$

w/ recursion relation:

$$A_n = \frac{c^2}{n}\left(A_{n-4} - A_{n-2}\right)$$ & $A_1 = 0$ from B.C. at $r = 0$

We also require $F = 0$ at $r^* = 1$

$$\therefore \sum A_n = A_0 \left\{ 1 - \frac{c^2}{4} + \frac{1}{16}\left(c^2 + \frac{c^4}{4}\right) - \dots \right\} = 0$$

gives eigen values c_i. Since of inf. degree, gives inf. # sol'ns

Numerically, first $c_i = 2.704$

From B.C. at $z^* = 0$, $T^* = 1$ thus:

$$T^* = \sum_{i=1}^{\infty} B_i F_i(r^*, c_i)\, e^{-c_i^2 z^*}$$

$$\text{where } B_i = \frac{\int_0^1 F_i(r^*)\, r^* (1 - r^{*2})\, dr^*}{\int_0^1 F_i^2(r^*)\, r^* (1 - r^{*2})\, dr^*}$$

which may be det. numerically

What is $T_b^*(z^*)$?

$$T_b^* = \frac{T_1 - T_b}{T_1 - T_0} = \frac{\int_0^1 T^* (1 - r^{*2})\, r^*\, dr^*}{\int_0^1 (1 - r^{*2})\, r^*\, dr^*}$$

$$= 4 \int_0^1 \sum_{i=1}^{\infty} B_i F_i\, e^{-c_i^2 z^*} (1 - r^{*2})\, r^*\, dr^*$$

$$= 4 \sum_{i=1}^{\infty} B_i e^{-c_i^2 z^*} \int_0^1 r^* (1 - r^{*2}) F_i(r^*)\, dr^*$$

from DE

$$= 4 \sum_{i=1}^{\infty} B_i e^{-c_i^2 z^*} \int_0^1 -\frac{1}{c_i^2} \frac{d}{dr^*}\left(r^* \frac{dF_i}{dr^*} \right) dr^*$$

$$= -4 \sum_{i=1}^{\infty} \frac{B_i e^{-c_i^2 z^*}}{c_i^2} \left(\frac{dF_i}{dr^*} \right)\Bigg|_{r^*=1}$$

which is what would be obtained from an energy balance!

$$h_{loc} \equiv \frac{k \frac{\partial T}{\partial r}\Big|_{r=R}}{T_1 - T_b} \quad \text{and} \quad Nu \equiv \frac{2Rh}{k}$$

$$\therefore (Nu)_{loc} = \frac{2R}{k} \frac{k \frac{\partial T}{\partial r}\Big|_{r=R}}{T_1 - T_b} = \frac{-2 \frac{\partial T^*}{\partial r^*}\Big|_{r^*=1}}{T_b^*}$$

$$= \frac{2 \sum_{i=1}^{\infty} B_i e^{-c_i^2 z^*} \left(\frac{dF_i}{dr^*} \right)_{r^*=1}}{4 \sum_{i=1}^{\infty} \frac{B_i e^{-c_i^2 z^*}}{c_i^2} \left(\frac{dF_i}{dr^*} \right)_{r^*=1}}$$

At large z^*, ($z^* > .25$) only leading term is important, thus

$$(Nu)_{loc} \cong \frac{c_1^2}{2} \cong 3.656 \quad \text{and} \quad T_b^* \approx 0.819\, e^{-7.3125 z^*}$$

Numerical Sol'n to S-L Probs

Run Ex 36 - Quenching of slab
(cheg 258 notes)

⇒ set up, then numerical sol'n

~~⇒ Then solve PDE as Time-dep. marching sol'n (numerical stability issues)~~

Sturm - Liouville Bdy Value Problems

$$[p(x)y']' - q(x)y + \lambda r(x) y = 0$$

eigenvalue

$a_1 y(0) + a_2 y'(0) = 0$

$b_1 y(1) + b_2 y'(1) = 0$

p, p', q & r are continuous

$p(x) > 0$ & $r(x) > 0$ for $x \in [0,1]$

Problem arises from sep. variables solution

How do we get eigenvalues & eigenvectors?

we have a problem of the form:

$$L[y] = \lambda r(x) y$$

Convert the problem to a set of finite dif. algebraic equations!

So:

$$[p(x)\,y']' \equiv \frac{[y_{k+1} - y_k]\,p(x_{k+\frac{1}{2}}) - [y_k - y_{k-1}]\,p(x_{k-\frac{1}{2}})}{h^2}$$

$$\frac{[p(x)\,y']'}{\cancel{r(x)}} - \frac{q(x)}{\cancel{r(x)}}\,y = -\lambda\, y\, r(x)$$

$$y_{k+1}\left\{\frac{1}{h^2}\,\frac{p(x_{k+\frac{1}{2}})}{\cancel{r(x_k)}}\right\} + y_k\left\{-\frac{\left(p(x_{k+\frac{1}{2}}) + p(x_{k-\frac{1}{2}})\right)}{h^2\,\cancel{r(x_k)}} - \frac{q(x_k)}{\cancel{r(x_k)}}\right\}$$

$$+\, y_{k-1}\left\{\frac{1}{h^2}\,\frac{p(x_{k-\frac{1}{2}})}{\cancel{r(x_k)}}\right\} = -\lambda\, y_k\, r(x_k)$$

with appropriate B.C.'s at $k = 0, N$

So: $\underset{\approx}{A}\;\underset{\sim}{y} = -\lambda\,\underset{\sim}{y}\cdot\underset{\approx}{r} \rightarrow$ diag matrix w/ elements $r(x_k)$

we can solve this problem using a modification of QR factorization:

Get N eigenvalues & N eigenvectors!

First few are accurate, later ones inaccurate

One approach:

Use matlab to solve finite dif. to get eigenvectors & eigenvalues (approx), then use shooting method to get a more accurate result!

Matlab gets you close, shooting method is more accurate!

```
clear
echo on
%This example demonstrates the solution of the
%Sturm Liouville eigenvalue problem via conversion
%of the differential equation into a set of
%algebraic equations.  The example uses the attached
%routine slsolve.m which requires that you define
%the functions p, q, and w as well as boundary
%conditions in standard Sturm Liouville form.  You
%can extract the necessary information by typing
%help slsolve:
pause
help slsolve
pause
%Let's apply this to the simple equation:
%
%  y'' + lambda y = 0
%
%with boundary conditions:
%
%  y(0)=0,   y'(1)=0
%
pause
%The solution to this equation is just:
%
%  y = sin ((n-.5)*pi*x),   n = 1, 2, 3, ...
%
%with the eigenvalues:
%
%  lambda = (n-.5)^2 * pi^2
pause
%For this problem the functions p, q, and w are:
%  p = 1
%  q = 0
%  w = 1
%which are particularly simple.  These have been
%stored in the function files pfun.m, qfun.m, and
%wfun.m
%
%The boundary conditions are described by the array:
%
%  bc = [1,0,0,1]
pause
%Let's work the example:
bc = [1,0,0,1];
n=50;
[lambda,eigenvec]=slsolve('pfun','qfun','wfun',bc,n);
%and that's all there is to it!
pause
%Let's examine these results.  First we list the first
%five eigenvalues:
lambda(1:5).^.5/pi
pause
%Note that the eigenvalues match what we expected pretty
%well, except that the error is increasing as we go to
%higher eigenvalues.  This is clearer if we plot the
%calculated eigenvalues against the exact values:
pause
i=[1:length(lambda)];
```

```
figure(1)
plot(i,i-.5,i,lambda.^.5/pi,'o')
xlabel('n')
ylabel('sqrt(lambda)/pi')
legend('exact value (n-0.5)','numerical value')
title('Comparison of Calculated and Exact Eigenvalues')
%Note that the values tend to fall off very badly for
%the largest eigenvalues!
pause
%Next we look at the eigenfunctions themselves.  We shall
%only examine the first five:
x=[0:1/n:1];
figure(2)
plot(x,eigenvec(:,1:5))
xlabel('x')
ylabel('y')
legend(num2str([1:5]'))
title('First Five Eigenfunctions')
%again we see that the solution starts to get a bit ragged
%for the higher eigenvectors.  Note that the leading eigen-
%vector has no nodes (zero crossings) in the interior of
%the domain, while the second has one, the third has two, and
%so forth.  This is typical of eigenvalue problems.  All the
%eigenvalues have a maximum value of 2^.5 because of the
%normalization: the average value of sin^2 over a period is
%just 1/2.
pause
%In conclusion, you may find this simple routine useful
%in dealing with Sturm-Liouville eigenvalue problems.  These
%differential equations are very common in transport problems
%such as the transient heating of a ball, cylinder, or slab,
%or the approach to an asymptotic velocity distribution in
%startup flow in a pipe.  It even arises in determining the
%frequency of a violin string!
echo off
```

```
function [lambda,eigenvec]=slsolve(p,q,w,bc,n)
%This function solves the Sturm-Liouville eigenvalue
%problem given by:
%
%  [p(x) y']' -q(x) y + lambda w(x) y = 0
%
%subject to the boundary conditions:
%
%  bc(1) y(0) + bc(2) y'(0) = 0
%  bc(3) y(1) + bc(4) y'(1) = 0
%
%over the domain 0< x <1.
%
%The function is called by the command:
%
%[lambda,eigenvec]=slsolve('pfun','qfun','wfun',bc,n);
%
%The function call requires that you provide the function
%names for the functions p, q, and w, as well as the boundary
%coefficients in the array bc.  The parameter n is the
```

```
%degree of discretization over the domain, e.g. h=1/n.
%The matrices A and W which are generated to solve the
%problem will thus be of size (n+1,n+1).  The function
%returns the eigenvalues in the array lambda (sorted by
%size in ascending order) and the matrix eigenvec which
%contains the corresponding eigenfunctions.  The eigenfunctions
%are all normalized so that the integral of the square times
%the weight function over the domain [0,1] is unity.
%
%A last note on error:  The code uses second order derivative
%approximations, so the error in the eigenvalues and eigen-
%vectors will be of O(1/n^2).  In general, the first few
%eigenvalues will be reliable, but the accuracy will deteriorate
%as you look at the higher eigenvalues, with the last few
%being meaningless.

h=1/n; %set discretization
x=[0:h:1]; %this is the array of x values

%Now we set up the arrays used in making the matrix A:
pp=zeros(1,n+1);
pm=zeros(1,n+1);
ww=zeros(1,n+1);
qq=zeros(1,n+1);
for i=2:n,
 pp(i)=feval(p,x(i)+h/2);
 pm(i)=feval(p,x(i)-h/2);
 ww(i)=feval(w,x(i));
 qq(i)=feval(q,x(i));
end

%The matrix W is easy:
weight=-diag(ww);

%The matrix A is a bit more complex.  First we do
%the main diagonal:
a=diag(-pp-pm-qq*h^2);
%and then the super and sub diagonals:
a=a+diag(pp(1:n),1);
a=a+diag(pm(2:n+1),-1);

%And now for the boundary conditions.  First at x=0:
a(1,1)=bc(1)-bc(2)*1.5/h;
a(1,2)=bc(2)*2/h;
a(1,3)=-bc(2)/2/h;

%and at x=1:
a(n+1,n+1)=bc(3)+bc(4)*1.5/h;
a(n+1,n)=-bc(4)*2/h;
a(n+1,n-1)=bc(4)/2/h;

%Finally, we divide by h^2:
a=a/h^2;

%Now we are ready to calculate the eigenvalues:
[v,d]=eig(a,weight);

%The number of eigenvalues and vectors will be less
%than the size of A and W, thus:
evals=diag(d);
```

```
i=find(finite(evals));
evals=evals(i);
evecs=v(:,i);

%Now we sort the eigenvalues and eigenvectors according
%to the size of the eigenvalues:
[temp,i]=sort(abs(real(evals)));
lambda=evals(i);
evecs=evecs(:,i);

%and finally, we normalize the eigenvectors so that the
%integral over the domain of the square of the eigenvector
%times the weight function equals unity. We use trapezoidal
%rule integration:
tw=diag(feval(w,x))/n;
tw(1,1)=tw(1,1)/2;
tw(n+1,n+1)=tw(n+1,n+1)/2;
enorm=abs(diag(evecs'*tw*evecs)).^.5;
for j=1:length(lambda),
 eigenvec(:,j)=evecs(:,j)/enorm(j)/sign(evecs(2,j));
end
```

```
function y=pfun(x)
%This is the p function in the Sturm-Liouville problem:
y=ones(size(x));
```

```
function y=qfun(x)
%This is the q function in the Sturm-Liouville problem:
y=zeros(size(x));
```

```
function y=wfun(x)
%This is the w function in the Sturm-Liouville problem:
y=ones(size(x));
```

Alt Sol'n tech –

PDE, marching forward in time

→ Easy

⇒ stability issues w/ matrix

Numerically unstable

Run example 31a

(Quenching of slab again)

```
clear
reset(0)
echo on
%Example 31a
%
%Numerical stability problems often occur in the
%solution of partial differential equations.  Here
%we illustrate this with the example of the quenching
%of a finite slab using a very simple numerical method.
%
%Consider a slab of width 2b initially at a temperature T1,
%and then quench the temperature at the surface to a value
%T0.  The thermal diffusivity alpha is assumed to be a
%constant.  We render x dimensionless with respect to the
%half-width b, and time dimensionless with respect to the
%diffusion time b^2/alpha.  Thus we have the equation:
%
% dT/dt = d2T/dx2; T(t=0)=1, T(x=1)=0, dT/dx(x=0)=0
%
%The last is the symmetry condition at the centerline.
pause
%OK, now we solve this as a system of first order ODE's.
%We discretize x:
n=10;
dx=1/n;
x=[0:dx:1];

%and we also discretize T:
T=ones(n+1,1);
pause
%Now we solve this as a function of time.  We shall use
%the Euler method for simplicity:
%
%We construct the tri-diagonal matrix governing the
%discretized equation:
a=-2*eye(n+1);
a=a+diag(ones(1,n),-1);
a=a+diag(ones(1,n),+1);
a=a/dx/dx;
%And we modify the first and last rows to account for the
%boundary conditions:
a(1,:)=zeros(1,n+1);
a(n+1,:)=zeros(1,n+1);
pause
%We can now integrate forward in time:
t=0;
dt=0.005;
T=ones(n+1,1);
T(n+1)=0;
figure(1)
plot(x,T)
drawnow
for k=1:1/dt
 T=T+dt*a*T;
 T(n+1)=0;              %This is the zero Temperature BC at x=1
 T(1)=(4*T(2)-T(3))/3; %This is the zero derivative BC at x=0
 plot(x,T)
 axis([0 1 0 1])
 xlabel('x')
```

```
 ylabel('T')
 t=t+dt;
 drawnow
 echo off
end
echo on
pause
%The integration procedure behaved very well.  Now let's do
%it again with a -very- slightly larger time step:
dt=0.0052;
pause
t=0;
T=ones(n+1,1);
T(n+1)=0;
figure(1)
plot(x,T)
drawnow
for k=1:1/dt
 T=T+dt*a*T;
 T(n+1)=0;
 T(1)=(4*T(2)-T(3))/3;
 plot(x,T)
 axis([0 1 0 1])
 xlabel('x')
 ylabel('T')
 t=t+dt;
 drawnow
 echo off
end
echo on
pause
%The dramatic difference between these two results illustrates
%the problem of numerical instability!  We require dt < 0.5 dx^2
%for the method to be stable.  Any explicit integration method
%would have the same stability limitations.  We can get around
%this limitation by using implicit integration techniques.
pause
%We finish this example by using the implicit Backward Euler Rule
%to integrate this problem forward.  First let's double dt:
dt=dt*2
%This would be *really* unstable for an explicit method!  Next
%we construct the matrices:
al=eye(n+1)-dt*a;
ar=eye(n+1);
pause
%We also have to incorporate the B.C.'s into the matrices:
al(1,:)=zeros(1,n+1);
al(1,1)=-3;
al(1,2)=4;
al(1,3)=-1;
ar(1,:)=zeros(1,n+1);
%and for the other B.C.
ar(n+1,:)=zeros(1,n+1);
%and we get the solution:
pause
t=0;
T=ones(n+1,1);
T(n+1)=0;
figure(1)
plot(x,T)
```

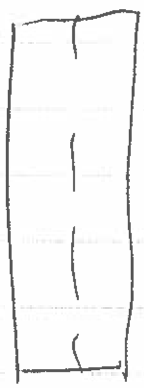
```
drawnow
for k=1:1/dt
 T=al\(ar*T);
 plot(x,T)
 axis([0 1 0 1])
 xlabel('x')
 ylabel('T')
 t=t+dt;
 drawnow
 echo off
end
echo on
pause
%Note that this is now stable even for much larger values of dt!
echo off
```

Alt Prob - not always S-L, but close -

Quenching w/ finite Res -

Not orthogonal eigenf$^{\underline{n}}$

Finite Reservoir



T_R

$$T\Big|_{t=0} = T_0$$

$$\frac{\partial T}{\partial y}\Big|_{y=0} = 0$$

$$\frac{\partial T}{\partial t}\Big|_{y=h} = Ak\frac{\partial T}{\partial y}\Big|_{y=h}\cdot\frac{2}{V\rho c_p} \qquad T\Big|_{\substack{y=h\\t=0}} = T_R$$

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial y^2}$$

$$T^* = \frac{T - T_\infty}{T_0 - T_\infty}$$

$$T_\infty = \frac{T_0(2hA) + T_R V}{2hA + V}$$

$$\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial y^{*2}}$$

$$y^* = y/h, \quad t^* = \frac{\alpha t}{h^2}$$

$$\therefore \frac{\partial T^*}{\partial y^*}\Big|_{y^*=0} = 0, \quad \frac{\alpha}{h^2}\frac{\partial T^*}{\partial t^*}\Big|_{y^*=1} = \frac{-2A\alpha}{Vh}\frac{\partial T^*}{\partial y^*}\Big|_{y^*=1}$$

②

So: $$\left.\frac{\partial T^*}{\partial t^*}\right|_{y^*=1} = -\frac{2Ah}{V}\left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=1}$$

$$\left.T^*\right|_{y^*=1} \to 0 \text{ as } t^* \to \infty$$

$$\left.T^*\right|_{t^*=0} = 1$$

Let $T^* = G(t^*)\,F(y^*)$

$$\therefore \quad \frac{G'}{G} = \frac{F''}{F} = -c^2$$

$$F'' + c^2 F = 0 \qquad G = e^{-c^2 t^*}$$

Now for B.C.'s:

$$\frac{G'}{G} = -\frac{2Ah}{V}\,\frac{F'(1)}{F(1)} = -c^2$$

$$\therefore \quad F'(1) - \frac{Vc^2}{2Ah}F(1) = 0$$

$$F'(0) = 0$$

not S-L form

Note: dif. once again to get a true S-L problem!

But not useful!

$$F = \cos c y$$

$$\therefore \; c \sin c - \frac{V}{2Ah} c^2 \cos c = 0$$

$$\frac{\tan c_n}{c_n} = + \frac{V}{2Ah}$$

so:

$$T^* = \sum_{n=1}^{\infty} A_n e^{-c_n^2 t^*} \cos c_n y^*$$

where: c_n s.t. $\dfrac{\tan c_n}{c_n} = \dfrac{V}{2Ah}$

Are eigenf$^{\underline{n}}$'s orthogonal?

$$\cancel{\int_0^1 \cos c_i y^* \cos c_j y^* \, dy^* =}$$

$$\int_0^1 F_i F_j \, dy^* = \int_0^1 -\frac{1}{c_i^2} F_i'' F_j \, dy = -\frac{1}{c_i^2} F_i' F_j \Big|_0^1$$

$$+ \int_0^1 + \frac{1}{c_i^2} F_i' F_j' \, dy^* = -\frac{1}{c_i^2}\left(F_i' F_j - F_i F_j'\right)\Big|_0^1 - \int_0^1 \frac{1}{c_i^2} F_i F_j'' \, d$$

$$= -\frac{1}{c_i^2}\left(F_i' F_j - F_i F_j'\right)\Big|_0^1 + \int_0^1 \frac{c_j^2}{c_i^2} F_i F_j \, dy$$

$$\therefore \int_0^1 F_i F_j \, dy = \frac{1}{1 - c_j^2/c_i^2} \frac{-1}{c_i^2} \left(F_i' F_j - F_i F_j' \right)_0^1$$

Now $F_i'(0) = F_j'(0) = 0$

$$\therefore \int_0^1 F_i F_j \, dy = \frac{1}{c_j^2 - c_i^2} \left(F_i'(1) F_j(1) - F_i(1) F_j'(1) \right)$$

$$F_i'(1) = \frac{V c_i^2}{2Ah} F_i(1)$$

$$\text{SO: } \int_0^1 F_i F_j \, dy = \frac{1}{c_j^2 - c_i^2} \left(\frac{V c_i^2}{2Ah} F_i(1) F_j(1) - \frac{V c_j^2}{2Ah} F_i(1) F_j(1) \right)$$

$$= -\frac{V}{2Ah} F_i(1) F_j(1) \neq 0 \, !$$

$$i \neq j$$

$$T^*\Big|_{t^*=0} = 1 = \sum_{n=1}^{\infty} A_n F_n(y^*)$$

Mult. by F_m:

$$\int_0^1 F_m \, dy^* = \sum_{n=1}^{\infty} A_n \int_0^1 F_n F_m \, dy^*$$

$$= \sum_{n \neq m} A_n \left(\frac{-V}{2Ah} \right) F_n(1) F_m(1)$$

$$+ A_m \int_0^1 F_m^2 \, dy^*$$

Choose:

$$F_n = \frac{\cos \sigma_n y^*}{\cos \sigma_n}$$

$\therefore F_n(1) = 1$

$$\therefore \int_0^1 F_m \, dy^* = \sum_{n=1}^{\infty} A_n \left(\frac{-V}{2Ah} \right)$$

$$+ A_m \left(\int_0^1 F_m^2 \, dy^* + \frac{V}{2Ah} \right)$$

$$\text{Now} \quad A_m = \frac{\int_0^1 F_m \, dy^* + \frac{V}{2Ah} \sum_{n=1}^{\infty} A_n}{\int_0^1 F_m^2 \, dy^* + \frac{V}{2Ah}}$$

$$\text{Let } \sum_{n=1}^{\infty} A_n = \lambda$$

$$\therefore A_m = \frac{\int_0^1 F_m \, dy^* + \frac{V}{2Ah} \lambda}{\int_0^1 F_m^2 \, dy^* + \frac{V}{2Ah}}$$

$$\lambda = \sum A_m = \sum_{m=1}^{\infty} \frac{\int_0^1 F_m \, dy^*}{\int_0^1 F_m^2 \, dy^* + \frac{V}{2Ah}} + \frac{V}{2Ah} \lambda \sum \frac{1}{\int_0^1 F_m^2 \, dy^*}$$

$$\text{So} \quad \lambda = \frac{\sum_{m=1}^{\infty} \frac{\int_0^1 F_m \, dy}{\int_0^1 F_m^2 \, dy^* + \frac{V}{2Ah}}}{1 - \frac{V}{2Ah} \sum_{m=1}^{\infty} \frac{1}{\int_0^1 F_m^2 \, dy^*}}$$

Similitude

Have introduced eq^n of E, studied some probs ⇒ now let's look at dimensional analysis

Rather than just dim. anal, look at lger subject of similitude of which dim. anal. is a part.

1) develop concepts (draw on fluid & solid mech as well as E transp) (1 wk)

2) Apply to E&M transp.

Ref: Barenblatt: similarity, self-similarity & intermediate asymptotics '79.

Also, Van Dyke notes

In any prob, have two quantities:

variables & parameters

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$x, y, \underset{\sim}{u}, T$ $\qquad\qquad$ U, F, a, L

may be <u>dependent</u> or <u>independent</u>

dep. variable: velocity, temp.

<u>indep</u> variable: position, time

indep. parameter: velocity, force, position ⇒ depends on how problem is formulated

In similitude, seek to reduce* <u>indep.</u> <u>var</u> & <u>param</u>: much better than reducing dep, i.e. stream f^n vs. sim. sol^n

Dimensional Analysis:
w/out eq'ns, write down # indep. param & variables all dep. param & variables depend on.

Nature knows no units: sol'n must be given in terms of dimensionless groups.

Ex: drag on sphere

$$F = f^n(\mu, U, a, \rho)$$

$$[F] = \frac{ML}{T^2},\ [\mu] = \frac{M}{LT},\ [U] = \frac{L}{T},\ [a] = L,\ [\rho] = \frac{M}{L^3}$$

Buckingham π thm: # dim. groups = # param − # indep. fundam. units

last may be det. from rank of matrix:

	F	ρ	U	a	μ
M	1	1	0	0	1
L	1	-3	1	1	-1
T	-2	0	-1	0	-1

has rank = 3 $\therefore$ 5 − 3 = 2 π groups

Choose $\frac{F}{\rho U^2 a^2}$, $\frac{U a \rho}{\mu}$

or $\dfrac{F}{\rho U^2 a^2} = f^n\left(\dfrac{U a \rho}{\mu}\right)$

thus $C_D = f^n(Re)$

Can often supply add'l info:

at low Re, expect inertia to become unimp

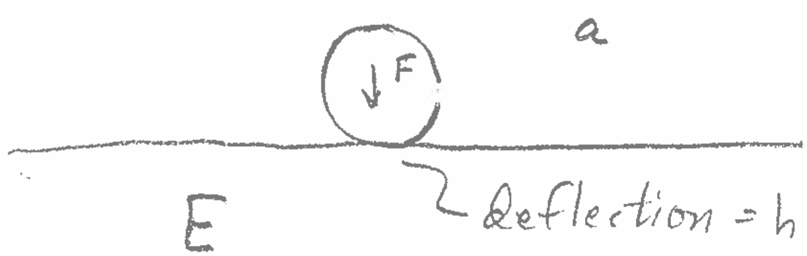
$\therefore \dfrac{F}{U\mu a} = cst$ as $Re \to 0$

↳ 6π (just 1 exp't)

End lec. 8

Rank of dim. matrix not always = # fundam units

ex: ball on plane



$\therefore h = f^n(E, F, a)$

$[h] = L, [a] = L, [E] = \dfrac{M}{LT^2}, [F] = \dfrac{ML}{T^2}$

$4 - 3 = 1 \quad \therefore \dfrac{h}{a} = cst??$

Form dim. matrix:

$$\begin{array}{c|c c}
 & a & \begin{matrix} h & E & F \end{matrix} \\
M & 0 & \left[\begin{matrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & -2 & -2 \end{matrix}\right] \\
L & 1 & \\
T & 0 &
\end{array}$$

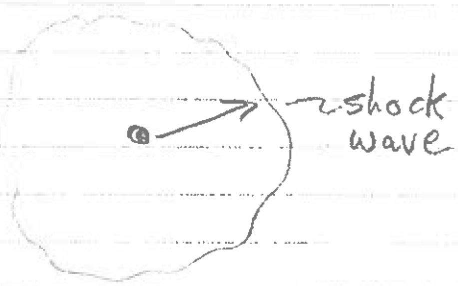
$\hookrightarrow$ rank $= 2$! (det. $= 0$)

$\therefore$ $4-2=2$ dim. groups:

$$\underline{\frac{h}{a} = f^{n}\left(\frac{F}{E a^2}\right)}$$

Occasionally, dim. anal. can be used to reduce # indep variables:

Ex: pt. explosion (G.I. Taylor in '40's)



$E, \rho_0, P_\infty, r, t$

E: Energy

r: radius of shock

wish to obtain $r(t)$ (dep variable)

$[E] = \frac{ML^2}{T^2}$, $[\rho_0] = \frac{M}{L^3}$, $[P_\infty] = \frac{M}{LT^2}$, $[r] = L$, $[t] = T$

$5-3=2$ dim. groups

Choose:

$$\frac{r\, P_\infty^{1/3}}{E^{1/3}} = f^{n}\left(\frac{t\, P_\infty^{5/6}}{\rho^{1/2}\, E^{1/3}}\right)$$

Not terrifically useful. Look at intermediate case: r lge enough that it looks like pt. explosion, but strong enough that

$$\frac{P}{P_\infty} \gg 1$$

$\therefore$ P_∞ won't matter (cold gas approx)

Thus 4-3=1 group, or:

$$\frac{r^5 \rho}{E\, t^2} = cst \qquad \text{or } r \sim t^{2/5}$$

constants from dim. anal. usually $O(1)$ $\Rightarrow$ often used as test of physical reasonableness of theory

In this case, cst = 1.03

Used by G I Taylor to est. E of atom bomb from movies. (1950)

Can often use add'l info to strengthen results, as above, but not always safe:

For flow past sphere, invoke $F \sim U$ for slow flow:

$$F \sim U \mu a$$

Now look at flow past cylinder of inf. extent:

$a,\ U,\ F/L,\ \mu,\ \rho \Big\}\ 5-3=2$

$$\therefore\ \frac{F/L}{U\mu} = f^{n}\left(\frac{Ua\rho}{\mu}\right)$$

Now at low Re, expect $F/L \neq f^{n}(\rho)$

$\therefore\ 4-3=1$ or $\dfrac{F/L}{U\mu} = cst$??!

not $f^{n}(a)$ which cannot be correct

Thus $\dfrac{F/L}{U\mu}$ must be $f^{n}(Re) \Rightarrow$ known

as Stokes paradox $\Rightarrow$ no solution to creeping flow eq'ns, could anticipate from dim. anal.

Actually, from Lamb:

$$F/L \cong 4\pi \frac{\mu U}{\log\left(4/Re\right) - \gamma + \frac{1}{2}} \quad \text{as } Re \to 0$$

$\hookrightarrow$ Euler's cst

May also strengthen dim. anal. by noticing that dimensional parameters only occur in certain comb.

Ex: Sphere falling thru fluid:

$U,\ a,\ \mu,\ \rho_f,\ \rho_s,\ g$