$6-3=3$

$\therefore$

$$U=\frac{\mu}{a\rho_\ell}f^{n}\left(\frac{\rho_s}{\rho_\ell},\ \frac{a^3\rho_\ell^2 g}{\mu^2}\right)$$

but at steady state, density of sphere can only affect motion thru $g(\rho_s-\rho_\ell)$

$\therefore$ $5-3=2$, or

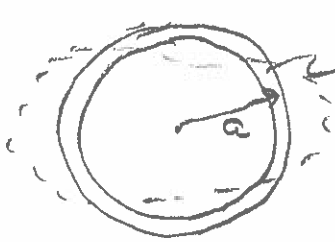
$$U=\frac{\mu}{a\rho_\ell}f^{n}\left(\frac{a^3\rho_\ell(\rho_s-\rho_\ell)g}{\mu^2}\right)$$

and at low Re, fluid inertia is unimp. so:

$$U=\underbrace{cst}_{\to\ \frac{9}{2}}\ \frac{a^2 g(\rho_s-\rho_\ell)}{\mu}$$

Inspectional Analysis: If you can write down eq'ns, may use dim. analysis to det. appropriate dim. groups ~~of~~ prob. depends on — often better than straight dim. anal.

Ex: mechanics of vibrating ring



c (circular cross-section, radius c)

what is the tone (dominant frequency of vibration)?

Look at vibration in plane, lowest mode

$$\omega = f^{\underline{n}}(a, c, \rho, E, \nu)$$

$E \to$ Young's mod.

$\nu \to$ poisson's modulus (dimensionless)

$\therefore 6-3=3$ or

$$\omega = \sqrt{\frac{E}{\rho a^2}}\, f^{\underline{n}}\left(\frac{c}{a}, \nu\right)$$

What is ω if $\frac{c}{a} \ll 1$? (Thin ring)

Answer cannot be obtained from dim. anal:

must look at eq'ns

Eq'n for displacement of pt. on a thin ring is given by:

$$\frac{E c^4}{4a^3}\left(\frac{\partial^6 x}{\partial\theta^6} + 2\frac{\partial^4 x}{\partial\theta^4} + \frac{\partial^2 x}{\partial\theta^2}\right) = \rho a c^2 \frac{\partial^2}{\partial t^2}\left(x - \frac{\partial^2 x}{\partial\theta^2}\right)$$

Note: does not depend on ν, often happens for slender rods & beams

Density & radius only appear in combination ρ/c^2

Thus: $\omega = cst\sqrt{\frac{E c^2}{\rho a^4}}$

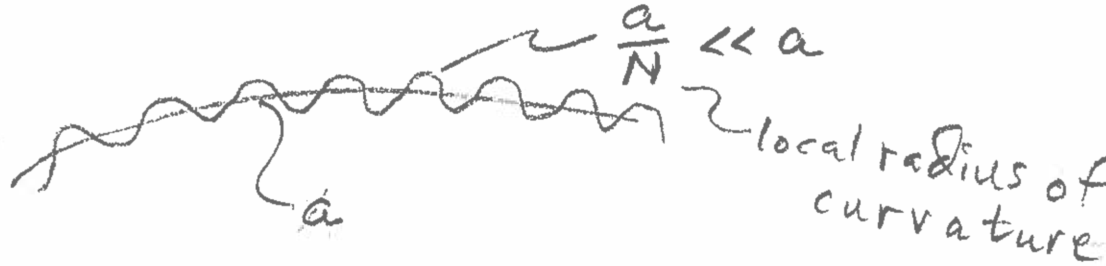
$cst = f^{\underline{n}}$ of which harmonic you look at (N)

For $N = 2$ (lowest) should be near unity

Solving eq$^{\underline{n}}$, find $cst = \frac{3}{\sqrt{5}} = 1.34$

What is behavior at large N?

Apply insight: for large N, tone should depend only on local curvature



Thus as $N \to \infty$, $\omega \sim cst\, N^2 \sqrt{\frac{E c^2}{\rho a^4}}$

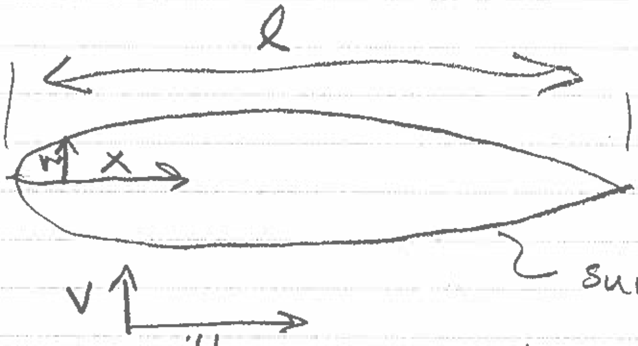
where we expect cst. to be $O(1) \Rightarrow$ actually is $\frac{1}{2}$

Exact sol$^{\underline{n}}$: $\omega = \frac{N(N^2-1)}{2\sqrt{N^2+1}} \sqrt{\frac{E c^2}{\rho a^4}}$

Most of this information recovered without solving the equation!

Inspectional Analysis: may be used to generalize results: New sol$^{\underline{n}}$s from old.

Ex: Subsonic, compressible flow past a slender body of revolution



surface: $\frac{r}{\ell} = \varepsilon T\left(\frac{x}{\ell}\right)$

where $\varepsilon \ll 1$ and $T = O(1)$

Inviscid flow, mach # M:

$$\frac{\underset{\sim}{u}}{U} = \underset{\sim}{f}\left(\frac{x}{\ell}, \frac{r}{\ell}; \varepsilon, M, \gamma\right)$$

γ ↳ adiabatic exponent

want to solve flow for all $M < 1$

Too much work to solve for <u>each</u> M, ∴ seek similarity between sol'ns at different M

Introduce perturbation velocity potential ϕ (irrotational flow)

s.t. $\underset{\sim}{u} = U \underset{\sim}{\nabla}(x + \phi)$

The eq'ns governing flow, to leading order in ε, are:

$$(1 - M^2)\phi_{xx} + \phi_{rr} + \frac{\phi_r}{r} = 0$$

$$\phi_r = \varepsilon T'(x) \quad \text{at } r = \varepsilon T(x) \quad \text{(tangential flow at surface)}$$

w/ $\phi_x, \phi_r \to 0$ as $x^2 + r^2 \to \infty$

Also renders problem indep. of γ

Let's stretch each variable independently
(may not have chosen best length scale for r, velocity for ϕ)

Let

$$\phi(x,r) = A\,\bar{\phi}(\bar{x},\bar{r}), \quad x = \bar{x}, \quad r = B\bar{r}$$

Plugging in, we obtain:

$$A(1-M^2)\,\bar{\phi}_{\bar{x}\bar{x}} + \frac{A}{B^2}\left(\bar{\phi}_{\bar{r}\bar{r}} + \frac{\bar{\phi}_{\bar{r}}}{\bar{r}}\right) = 0$$

or $$B^2(1-M^2)\,\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{r}\bar{r}} + \frac{\bar{\phi}_{\bar{r}}}{\bar{r}} = 0$$

w/ B.C.'s:

$$\bar{\phi}_{\bar{r}} = \frac{B\varepsilon}{A}\,T'(\bar{x}) \quad \text{at } \bar{r} = \frac{\varepsilon}{B}\,T(\bar{x})$$

$$\bar{\phi}_{\bar{x}}, \bar{\phi}_{\bar{r}} \to 0 \quad \text{as } \bar{x}^2 + \bar{r}^2 \to \infty$$

We thus have 3 combinations of ε, M, A & B:

$$B^2(1-M^2), \quad \frac{B\varepsilon}{A}, \quad \frac{\varepsilon}{B}$$

We may choose A, B any number of ways, but some are more convenient than others.

Choose A, B so as to eliminate M:

$$B^2(1-M^2)=1 \quad \text{or} \quad B=\frac{1}{(1-M^2)^{1/2}}$$

Let's also preserve tangent flow condition:

$$\therefore \frac{B}{A}=\frac{1}{B} \quad \text{or} \quad A=B^2=\frac{1}{1-M^2}$$

Thus we obtain the reduced problem:

$$\bar{\phi}_{\bar{x}\bar{x}}+\bar{\phi}_{\bar{r}\bar{r}}+\frac{\bar{\phi}_{\bar{r}}}{\bar{r}}=0$$

$$\bar{\phi}_{\bar{r}}=\bar{\varepsilon}\,T'(\bar{x}) \quad \text{at } \bar{r}=\bar{\varepsilon}\,T(\bar{x})$$

$$\bar{\phi}_{\bar{x}}, \bar{\phi}_{\bar{r}} \rightarrow 0 \quad \text{as } \bar{x}^2+\bar{r}^2 \rightarrow \infty$$

reduced # param. by one!

w/ similarity rule:

$$\phi(x, r; \varepsilon, M)=\frac{1}{1-M^2}\,\bar{\phi}\left(x, \frac{r}{\sqrt{1-M^2}}; \frac{\varepsilon}{\sqrt{1-M^2}}, 1\right)$$

Thus linearized flow past slender body of revolution may be related to an equivalent incompressible flow past a thinner body where all dimensions normal to dir. of flow have been contracted by $(1-M^2)^{1/2}$

End 9 lec.

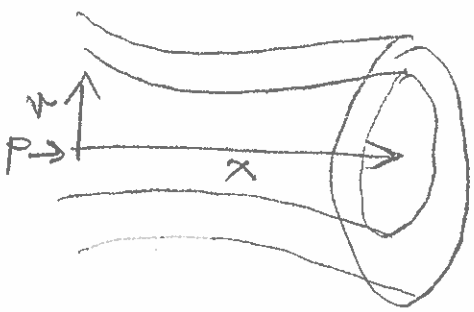
Has been extended to gen. 3-D slender shapes in supersonic & transonic flows.
Assoc. w/ Göthert (1940).

Self-similarity from Dim. Anal.

In last problem, stretching revealed hidden symmetry that permitted reduction of # indep. dim. param. on which prob. depends on. May also be used to reduce # indep var. on which prob. depends.

Ex: submerged laminar jet

viscous liquid, jet of same fluid emerging from small orifice ⇒ pt. source of momentum



ρ, ν

$P \to \frac{\text{momentum}}{\text{time}}$

problem is axisymmetric, if stable

$$\begin{array}{c|ccc|ccc}
 & u = f^n(& x, & r; & P, & \rho, & \nu) \\
L & 1 & 1 & 1 & 1 & -3 & 2 \\
T & -1 & 0 & 0 & -2 & 0 & -1 \\
M & 0 & 0 & 0 & 1 & 1 & 0
\end{array}$$

rank = 2

No ref. length can be formed from parameters,

thus: $u = \dfrac{\nu}{x} f^n\left(\dfrac{r}{x}; \dfrac{\sqrt{P}}{\sqrt{\rho}\,\nu}\right)$

Re of flow in orifice leading to jet

must relate r to x

2-D problem, so define stream $f^{\underline{n}}$:

$$u = \frac{\psi_r}{r}, \quad v = -\frac{\psi_x}{r}$$

$\therefore$ ψ has self-similar form:

$$\psi = \nu x \, f\left(\frac{r}{x}; Re\right)$$

This may be substituted into N-S eq'ns: get messy non-linear O.D.E, actually has simple sol'n:

$$\psi = \nu x \left(\frac{2(1-\zeta^2)}{(1+a)\zeta - \zeta^2}\right), \quad \zeta = \frac{x}{\sqrt{x^2+r^2}} \equiv \left(1 + \frac{r^2}{x^2}\right)^{-\frac{1}{2}}$$

where a is related to Re by a transcendental eq'n. <u>Very</u> rare to find exact sol'n to N-S eq'ns.

Similitude & self-similarity rarely found by dim. anal. alone: E.g. planar jet

Momentum P/L, ρ, ν $\Rightarrow$ can form ref. length,

thus: $$u = \frac{P/L}{\rho \nu} f^{\underline{n}}\left(\frac{Px}{L\rho\nu^2}, \frac{Py}{L\rho\nu^2}\right)$$

No self-similarity is revealed.

Problem <u>does</u> admit self-similar sol'n, but must obtain thru inspectional anal.

Self-Sim. from inspectional analysis:

Results from exceptionally symmetric structure of problem. How do we find it?

Morgan (1952) proposed 2 rules:

1) If a well defined prob. is invariant under a one-parameter continuous group of transf. of indep & dep. variables, the number of indep. var. may be reduced by one.

2) Reduction accomplished by introducing as new independent and dependent var. comb. that are invariant under transf.

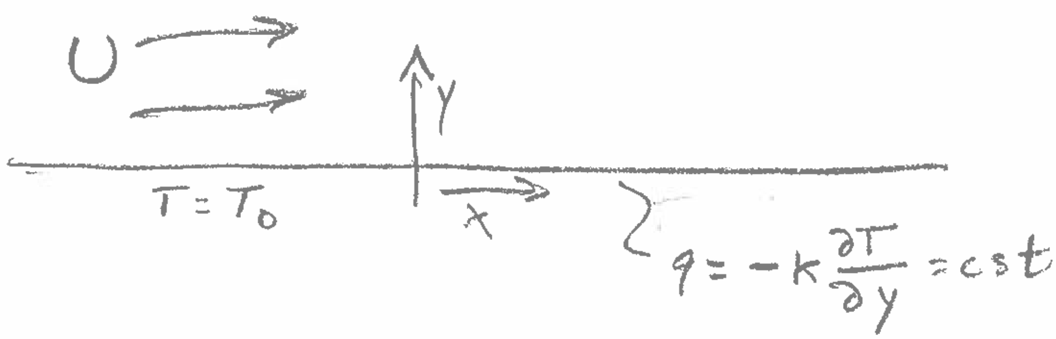
Ex: translation of origin. If prob. is invariant to continuous transf. $t = \bar{t} + C$, just elim. t

Note: C is continuous, not discrete!

Very powerful, but choice not always obvious.

We will restrict ourselves to simple stretching, i.e. $x = A\bar{x}$, etc.

Ex: Forced convection past a heated wall



Assume constant properties:

$$\rho \hat{C}_p U \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial y^2} \Rightarrow \text{we have neglected conduction in x-direction}$$

thus $\dfrac{\partial T}{\partial x} = \dfrac{\alpha}{U} \dfrac{\partial^2 T}{\partial y^2}$

we have a length scale (α/U) in this problem, thus let:

$$x^* = x\left(U/\alpha\right), \quad y^* = y\left(U/\alpha\right), \quad T^* = \frac{T - T_0}{\frac{\dot{q}\alpha}{UK}}$$

$$\therefore \frac{\partial T^*}{\partial x^*} = \frac{\partial^2 T^*}{\partial y^{*2}}, \quad T^* = 0 \begin{cases} y^* = 0, \ x^* = 0 \\ y^* \to \infty \end{cases}$$

$$\text{and } \frac{\partial T^*}{\partial y^*} = -1, \quad y^* = 0, \ x^* > 0$$

How do we solve this? Try stretching:

let $T^* = A\bar{T}$, $x^* = B\bar{x}$, $y^* = C\bar{y}$

$$\text{Thus } \frac{A}{B}\frac{\partial \bar{T}}{\partial \bar{x}} = \frac{A}{C^2}\frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \equiv \frac{\partial \bar{T}}{\partial \bar{x}} = \frac{B}{C^2}\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}$$

$$\text{w/ B.C.'s} \quad A\bar{T} = 0 \begin{cases} C\bar{y} = 0, \ B\bar{x} = 0 \\ C\bar{y} \to \infty \end{cases}$$

$$\text{and } \frac{A}{C}\frac{\partial \bar{T}}{\partial \bar{y}} = -1, \quad C\bar{y} = 0, \ B\bar{x} > 0$$

Have 3 free wishes (A, B, C) & 2 restrictions for invariance; $B/C^2 = 1$, $\frac{A}{C} = 1$

∴ invariant to <u>one</u> <u>param</u> group of transf. ; will admit sim. sol'n

Choose $\frac{B}{C^2} = 1 \equiv \frac{C}{B^{1/2}} = 1$ or $\eta = \frac{y}{x^{1/2}}$ as <u>sim. variable</u> : put time-like variable in denom. for <u>canonical form</u> ; y appears as higher deriv. in eq'n ⇒ $\eta \sim y$ simplifies algebra.

Now for similarity rule:

$$\frac{A}{C} = 1 \equiv \frac{A}{B^{1/2}} = 1 \quad \therefore \text{ let } T^* = x^{1/2} f(\eta)$$

Again, using canonical form

Thus : $T^* = x^{*1/2} f(\eta)$, $\eta = \frac{y^*}{x^{*1/2}}$

Now plug back into P.D.E. :

$$\frac{\partial T^*}{\partial y^*} = \frac{\partial}{\partial y^*}\left(x^{*1/2} f(\eta)\right) = x^{*1/2} \frac{\partial f}{\partial \eta} \overbrace{\frac{\partial \eta}{\partial y}}^{\sim \frac{1}{x^{1/2}}} = \frac{\partial f}{\partial \eta}$$

$$\frac{\partial^2 T^*}{\partial y^{*2}} = f'' \frac{1}{x^{*1/2}} \qquad \frac{\partial \eta}{\partial x^*} = -\frac{1}{2} \frac{y^*}{x^{*3/2}} = -\frac{1}{2} \frac{1}{x^*} \eta$$

and:

$$\frac{\partial T^*}{\partial x^*} = \frac{\partial\, x^{*\,1/2} f(\eta)}{\partial x^*} = \frac{1}{2} x^{-1/2} f - \frac{1}{2}\eta x^{-1/2} f'$$

Thus the PDE becomes:

$$\frac{1}{2} x^{*\,-1/2}(f - \eta f') = x^{*\,-1/2} f''$$

$$\text{or} \quad f'' + \frac{1}{2}\eta f' - \frac{1}{2} f = 0$$

Diff. to get integrable form!

$$\boxed{f''' = -\frac{1}{2}\eta f''}$$

$$f'' = c_1 e^{-\eta^2/4}$$

w/ B.C. $f(\infty) = 0$, $f'(0) = -1$

simple linear O.D.E. w/ 2 B.C.'s!

This problem has the solution:

$$f = 2\,\mathrm{ierfc}\left(\frac{\eta}{2}\right)$$

where ierfc ≡ integral complimentary error function

$$\mathrm{ierfc}(x) = \int_x^\infty (1 - \mathrm{erf}(x))\,dx$$

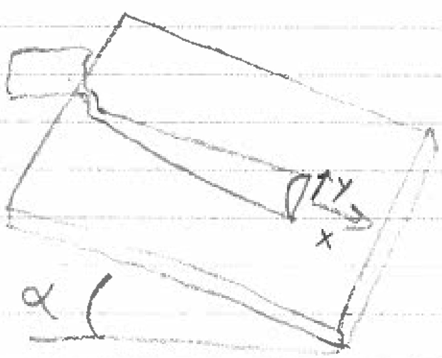
$$\text{Thus } T - T_0 = \frac{q}{k}\left(\frac{4x\alpha}{U}\right)^{1/2} \mathrm{ierfc}\left(\frac{y}{\left(\frac{4x\alpha}{U}\right)^{1/2}}\right)$$

Last was very simple problem: self similarity could have been revealed by inspection of eq^ns & physical insight: balancing y-cond w/ x-conv.

Method here provides rigorous way of getting profile.

Look at more complex problem:

Flow of viscous thread down plane:



if we neglect surface tension & inertia & are far enough downstream that stream is shallow & narrow then:

$$h\,h_{yy} + 3h_y^2 - 3\tan\alpha\, h_x = 0$$

w/ conservation of mass condition:

$$\frac{2}{3}\frac{g}{\nu}\sin\alpha \int_0^{y_e(x)} h^3(x,y)\,dy = Q$$

where stream edges are at $y = \pm y_e(x)$

Non-linear PDE: a real mess!

Try stretching: use same stretching

$$h = A\bar{h}, \quad x = B\bar{x}, \quad y = C\bar{y}, \quad y_e = C\bar{y}_e$$

Thus:

$$\frac{A^2}{C^2}\bar{h}\,\bar{h}_{\bar{y}\bar{y}} + 3\frac{A^2}{C^2}\bar{h}_{\bar{y}}^{\,2} - 3\frac{A}{B}\tan\alpha\,\bar{h}_{\bar{x}} = 0$$

or $$\bar{h}\,\bar{h}_{\bar{y}\bar{y}} + 3\bar{h}_{\bar{y}}^{\,2} - 3\frac{C^2}{AB}\tan\alpha\,\bar{h}_{\bar{x}} = 0$$

w/ B.C.'s $\bar{h} = 0$ at $\bar{y} = \pm\bar{y}_e$

& $$A^3 C\,\frac{2}{3}\frac{g}{\nu}\sin\alpha\int_0^{\bar{y}_e}\bar{h}^3\,d\bar{y} = Q$$

$\therefore$ invariant for $A^3C = \frac{C^2}{AB} = 1$

or $A = \frac{C^2}{B}$ $\therefore$ $\frac{C^7}{B^3} = 1 = \frac{B}{C^{3/7}}$, $AC^{1/7} = 1$

End lec. 10

thus $h = x^{-1/7} f(\zeta)$, $\zeta = \frac{y}{x^{3/7}}$

Substituting in:

$$h_y = x^{-1/7}\frac{\partial f}{\partial\zeta}\frac{\partial\zeta}{\partial y} = x^{-4/7} f'$$

$$h_{yy} = \frac{1}{x} f''$$

$$h_x = \frac{\partial}{\partial x}\left(x^{-1/7} f(\zeta)\right) = -\frac{1}{7}x^{-8/7} f - \frac{3}{7}\frac{y}{x^{11/7}} f'$$

$$= -\frac{1}{7}x^{-8/7}\left(f + 3\zeta f'\right)$$

thus

$$x^{-8/7} f f'' + 3 x^{-8/7} f'^2 + \frac{3}{7} \tan\alpha \, x^{-8/7} (f + 3\zeta f') = 0$$

or $f f'' + 3 f'^2 + \frac{3}{7} \tan\alpha (f + 3\zeta f') = 0$

multiplying by f^2:

$$f^3 f'' + 3 f^2 f'^2 + \frac{3}{7} \tan\alpha (f^3 + 3\zeta f^2 f') = 0$$

$$\equiv \frac{d}{d\zeta}\left(f^3 f' + \frac{3}{7} \tan\alpha \, \zeta f^3 \right) = 0$$

or $\frac{d}{d\zeta}\left[f^3 \left(f' + \frac{3}{7} \tan\alpha \, \zeta \right) \right] = 0$

w/ B.C. $f = 0$ at $\zeta = \pm \zeta_e$ (cst)

and $\int_0^{\zeta_e} f^3 \, d\zeta = \frac{3}{2} \frac{\nu Q}{g \sin\alpha}$

Integrating once: finite at edges

$$f^3 \left(f' + \frac{3}{7} \tan\alpha \, \zeta \right) = C_1$$

Now at $\zeta = \pm \zeta_e$ $f = 0$ $\therefore C_1 = 0$

thus $f' = -\frac{3}{7} \tan\alpha \, \zeta$

hence $f = C_2 - \frac{3}{14}\tan\alpha\,\zeta^2$

$$= \frac{3}{14}\tan\alpha\,\zeta_e^2\left(1 - \left(\frac{\zeta}{\zeta_e}\right)^2\right)$$

parabolic profile !

What is ζ_e ?

$$\int_0^1 \zeta_e \left(\frac{3}{14}\tan\alpha\,\zeta_e^2\right)^3 (1-\zeta'^2)^3\,d\zeta' = \frac{3}{2}\,\frac{\nu Q}{g\sin\alpha}$$

where $\zeta' = \zeta/\zeta_e$

Thus $\zeta_e = \left[\dfrac{5\cdot 7^4}{4\cdot 3^2}\,\dfrac{Q\nu}{g\sin\alpha\tan^3\alpha}\right]^{1/7}$

and $y_e = \left[\dfrac{5\cdot 7^4}{4\cdot 3^2}\,\dfrac{Q\nu x^3}{g\sin\alpha\tan^3\alpha}\right]^{1/7}$

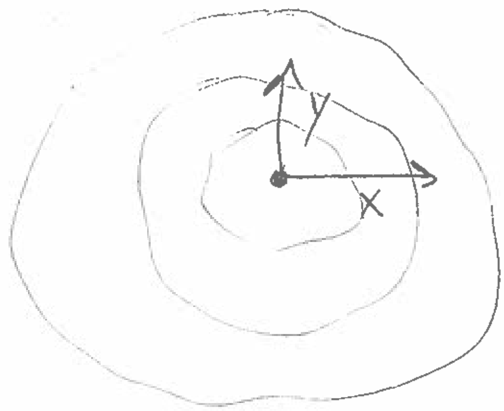
Ref: P.C. Smith, 1973 JFM

~~Stretching does not always work, although it gets most self-sim~~

~~Ex: periodic heating of semi-infinite slab (i.e., frost line in earth's crust)~~

~~(will discuss in more detail later unsteady cond.)~~

Diffusion from a point source



$$c\Big|_{t=0} = \delta(x)\,\delta(y)$$

Dirac δ f$^{\underline{n}}$: $\delta(y) = 0 \quad y \neq 0$

δ has units 1/length $\displaystyle\int_{-\varepsilon}^{\varepsilon} \delta(y)\,dy = 1$

$$\frac{\partial c}{\partial t} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right)$$

diff$^{\underline{n}}$ is a random walk process. Rnd walk in each direction is <u>indep</u> $\therefore$ let:

$c = f(x,t)\, g(y,t)$ where

$$\frac{\partial g}{\partial t} = D\left(\frac{\partial^2 g}{\partial y^2}\right) \qquad g(y,t)\Big|_{t=0} = \delta(y)$$

Let's apply dim'l analysis:

Only param. is D ⇒ <u>no</u> ref length!

∴ we get

$$g^* = g(Dt)^{1/2}$$

units 1/length

and $\xi = \frac{y}{(Dt)^{1/2}}$

so: $g = \frac{g^*}{(Dt)^{1/2}}$

Plugging in:

$$\frac{\partial^2 g}{\partial y^2} = \frac{g^{*''}}{(Dt)^{1/2}}\frac{1}{Dt} \qquad \frac{\partial g}{\partial t} = \frac{-1}{2}\frac{g}{(Dt)^{1/2}}\frac{1}{t} - \frac{1}{2}\frac{\xi}{(Dt)^{1/2}}\frac{1}{t}g'$$

so $-\frac{1}{2}(g^* + \xi g^{*'}) = g^{*''}$

or $g^{*''} = -\frac{1}{2}(\xi g^*)'$

so $g^{*'} = -\frac{1}{2}\xi g^* + C$

but as $\xi \to \infty$ $g^* \to 0$ ∴ $C = 0$

So

$$g^{*\prime} = -\frac{1}{2}\xi g^{*}$$

$$g^{*} = c_2 e^{-\frac{\xi^2}{4}}$$

where C_2 det. from $\int_{-\infty}^{\infty} g^{*} d\xi = 1$

or $C_2 = \frac{1}{\sqrt{4\pi}}$

$$\therefore \quad g = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{y^2}{4Dt}}$$

for one-D diffusion

Now let's substitute back into the original D.E.:

$$\frac{\partial c}{\partial t} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right)$$

$$c = f(x, y, t)\, g(y, t)$$

Thus

$$g\frac{\partial f}{\partial t} + f\frac{\partial g}{\partial t} = D\, g\frac{\partial^2 f}{\partial x^2} + D\left(g\frac{\partial^2 f}{\partial y^2} + 2\frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial y^2}\right)$$

not $f^{n}(x)$

D

Rearranging, we obtain:

$$f\left(\cancel{\frac{\partial g}{\partial t} - D\frac{\partial^2 g}{\partial y^2}}\right) + g\left(\frac{\partial f}{\partial t} - D\frac{\partial^2 f}{\partial x^2}\right) = D\left(g\frac{\partial^2 f}{\partial y^2} + 2\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right)$$

0 from D.E.

Now from the solution for $g(y,t)$ we have:

$$\frac{\partial g}{\partial y} = \frac{-2y}{4Dt}\, g$$

Thus, dividing by g, we get:

$$\frac{\partial f}{\partial t} + \frac{y}{t}\frac{\partial f}{\partial y} = D\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

with $f\Big|_{t=0} = \delta(x)$

For this case we have a sol'n to the above problem if $f \neq f^n(y)$

Thus $\frac{\partial f}{\partial y} = 0 = \frac{\partial^2 f}{\partial y^2}$

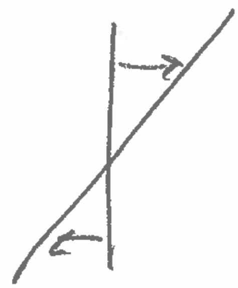
and $\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$, $f\Big|_{t=0} = \delta(x)$

or

$$f = \frac{1}{\sqrt{4\pi Dt}}\, e^{-\frac{x^2}{4Dt}}$$

hence $C(x,y,t) = \frac{1}{4\pi Dt}\, e^{-\frac{x^2+y^2}{4Dt}} = \frac{1}{4\pi Dt}\, e^{-\frac{r^2}{4Dt}}$

Now let's examine the case of a <u>steady</u> shear flow



$$\underset{\sim}{u} = u\,\underset{\sim}{e}_x \quad \text{where } u = \dot{\gamma} y$$

So: $$\frac{\partial c}{\partial t} + \dot{\gamma} y \frac{\partial c}{\partial x} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right)$$

There is no motion in the y direction, so dispersion in y direction will be unaffected by the shear flow. Thus let:

$c = g(y,t)\, f(x,y,t)$ (now f <u>will</u> be $f^{\underline{n}}(y)$)

where

$$\frac{\partial g}{\partial t} = D\frac{\partial^2 g}{\partial y^2}, \quad g\big|_{t=0} = \delta(y)$$

Plugging back in we get:

$$\frac{\partial f}{\partial t} + \dot{\gamma} y \frac{\partial f}{\partial x} + \frac{y}{t}\frac{\partial f}{\partial y} = D\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

$$f\big|_{t=0} = \delta(x)$$

F

We seek a sol'n of the form:

$f(x, y; t) = f(\zeta, t)$ where:

$$\zeta = x - \underbrace{\lambda \dot{\gamma} y t}_{\text{shift by strain in } x \text{ direction}}$$

Thus:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \zeta}, \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \zeta}(-\lambda \dot{\gamma} t)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial \zeta^2}(\lambda^2 \dot{\gamma}^2 t^2)$$

$$\left(\frac{\partial f}{\partial t}\right)_{x,y} = \left(\frac{\partial f}{\partial t}\right)_{\zeta} + \frac{\partial f}{\partial \zeta}(-\lambda \dot{\gamma} y)$$

So:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \zeta}\underbrace{\left(-\lambda \dot{\gamma} y + \dot{\gamma} y - \lambda \dot{\gamma} t \frac{y}{t}\right)}_{= \dot{\gamma} y (1 - 2\lambda)} = D(1 + \lambda^2 \dot{\gamma}^2 t^2)\frac{\partial^2 f}{\partial \zeta^2}$$

problem will not be a $f^{\underline{n}}(y)$ if $\lambda = \frac{1}{2}$

Physically, similarity variable:

$$\zeta = x - \tfrac{1}{2}\dot{\gamma} t y$$

is disp. by average strain of molecule migrating from origin to y (e.g. avg. vel. during migration is $\tfrac{1}{2}\dot{\gamma} y$ over duration of time t)

So: $\dfrac{\partial f}{\partial t} = D\left(1 + \lambda^2 \dot{\gamma}^2 t^2\right)\dfrac{\partial^2 f}{\partial \zeta^2}$

where $f\Big|_{t=0} = \delta(\zeta)$

Let's define some new time variable ξ s.t. we get the one-D diffusion eq'n:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \xi}\frac{d\xi}{dt} = D\left(1 + \lambda^2 \dot{\gamma}^2 t^2\right)\frac{\partial^2 f}{\partial \zeta^2}$$

so pick ξ s.t. $\dfrac{d\xi}{dt} = \left(1 + \lambda^2 \dot{\gamma}^2 t^2\right)$

$\therefore\ \xi = t + \tfrac{1}{3}\lambda^2 \dot{\gamma}^2 t^3 + \cancel{c} \to 0$ so that $\xi\Big|_{t=0} = 0$

hence $\dfrac{\partial f}{\partial \xi} = D\dfrac{\partial^2 f}{\partial \zeta^2}$ $\quad f\Big|_{\xi=0} = \delta(\zeta)$

So we have the solution:

$$f = \frac{1}{\sqrt{4\pi D \xi}}\, e^{-\frac{z^2}{4D\xi}}$$

or:

$$c = \frac{1}{4\pi D t\left(1+\frac{1}{12}\dot{\gamma}^2 t^2\right)^{1/2}}\, e^{-\frac{y^2}{4Dt}}\, e^{-\frac{\left(x-\frac{1}{2}\dot{\gamma} y t\right)^2}{4Dt\left(1+\frac{1}{12}\dot{\gamma}^2 t^2\right)}}$$

One question is what is the contribution of shear to the variance in the x direction?

$$\sigma_x^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2\, c\, dx\, dy$$

$$= 2Dt\left(1+\frac{1}{3}\dot{\gamma}^2 t^2\right)$$

so the effect is $\frac{1}{3}\dot{\gamma}^2 t^2 \Rightarrow$ as t grows variance is <u>dominated</u> by shear flow.

variance of dist in x-direction

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2/\pi D t}}\, x^2 e^{-\frac{x^2}{4Dt}}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{x\,(2Dt)}{\sqrt{4\pi D t}}\left(\frac{2x}{4Dt}\right) e^{-\frac{x^2}{4Dt}}\, dx$$

$$= -\frac{2Dt\, x}{\sqrt{4\pi D t}}\, e^{-\frac{x^2}{4Dt}}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{2Dt}{\sqrt{4\pi D t}}\, e^{-\frac{x^2}{4Dt}}\, dx$$

$$= 2Dt$$

$$\int_{-\infty}^{\infty} \frac{x^2}{\sqrt{4\pi D \xi}}\, e^{-\frac{(x - \frac{1}{2}\dot{\gamma} t y)^2}{4D\xi}}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{(z + \frac{1}{2}\dot{\gamma} t y)^2}{\sqrt{4\pi D \xi}}\, e^{-\frac{z^2}{4D\xi}}\, dz$$

$$= \int_{-\infty}^{\infty} \frac{z^2}{\sqrt{4\pi D\xi}}\, e^{-\frac{z^2}{4D\xi}}\, dz + \underbrace{\cancel{\int_{-\infty}^{\infty} \frac{z\,\dot{\gamma}\, t\, y}{\sqrt{4\pi D\xi}}\, e^{-\frac{z^2}{4D\xi}}\, dz}}_{0} + \int_{-\infty}^{\infty} \frac{\frac{1}{4}\dot{\gamma}^2 t^2 y^2}{\sqrt{4\pi D\xi}}$$

H 2

$$= 2D\xi + \frac{1}{4}\dot{\gamma}^2 t^2 y^2$$

integ. over y:

$$\sigma_x^2 = 2D\xi + \frac{1}{4}\dot{\gamma}^2 t^2 (2Dt)$$

$$= 2Dt\left(\frac{1}{4}\dot{\gamma}^2 t^2 + 1 + \frac{1}{12}\dot{\gamma}^2 t^2\right)$$

$$= 2Dt\left(1 + \frac{1}{3}\dot{\gamma}^2 t^2\right)$$

Now for the case where the shear flow is time dependent.

Let $$u = \underbrace{s(t)}_{\text{arb time dep. fn}}\ \underbrace{\dot{\gamma}}_{\text{char. mag.}}\ y$$

$$\therefore \frac{\partial c}{\partial t} + s(t)\,\dot{\gamma}\, y \frac{\partial c}{\partial x} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right)$$

Now again $c = fg$

where:
$$g = \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{y^2}{4Dt}}$$

and we have:

$$\frac{\partial f}{\partial t} + \dot{\gamma} y \frac{\partial f}{\partial x} + \frac{y}{t}\frac{\partial f}{\partial y} = D\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

$$f\Big|_{t=0} = \delta(x)$$

Let's take $f(x, y, t) = f(\eta, t)$

where $\eta = x - \dot{\gamma} y\, h(t)$

where $h(t)$ remains to be determined.

J

So:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\ ; \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\left(-\dot{\gamma} h\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \dot{\gamma}^2 h^2 \frac{\partial^2 f}{\partial \xi^2}$$

$$\left(\frac{\partial f}{\partial t}\right)_{x,y} = \left(\frac{\partial f}{\partial t}\right)_{\xi} + \frac{\partial f}{\partial \xi}\left(-\dot{\gamma} y h'\right)$$

So:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \xi}\Big(\underbrace{\dot{\gamma} y s - \dot{\gamma} y \frac{h}{t} - \dot{\gamma} y h'}_{=\ \dot{\gamma} y \left(s - \frac{h}{t} - h'\right)}\Big) = D\left(1 + \dot{\gamma}^2 h^2\right)\frac{\partial^2 f}{\partial \xi^2}$$

so problem is indep. of y if:

$$h' + \frac{h}{t} - s = 0 \quad , \quad h(0) = 0$$

This is a 1st order O.D.E. where:

$$h(t) = \frac{1}{t}\int_0^t t'\, s(t')\, dt'$$

So: $\displaystyle \frac{\partial f}{\partial t} = D\left(1 + \dot{\gamma}^2 h^2\right)\frac{\partial^2 f}{\partial \xi^2}$

K

Now we may define a new time variable ξ s.t.

$$\frac{\partial \xi}{\partial t} = (1 + \dot{\gamma}^2 h^2)$$

$$\xi = \int_0^t (1 + \dot{\gamma}^2 h^2)\, dt'$$

and $f = \frac{1}{\sqrt{4\pi D\xi}}\, e^{-\frac{z^2}{4D\xi}}$ as before!

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DIFFUSION FROM AN INITIAL POINT DISTRIBUTION IN AN UNBOUNDED OSCILLATING SIMPLE SHEAR FLOW

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Abstract—A self-similar closed form solution for diffusion from an initial point distribution of an infinitely dilute solute in an unbounded oscillating simple shear flow is presented. The solution is shown to agree with the well-known solution for diffusion in an unbounded steady simple shear flow in the correct limit, and also to reproduce the high frequency limiting dispersivity for oscillatory flow in closed conduits at large Schmidt numbers. The result is further generalized for arbitrary time dependent simple shear flows and source distributions.

1. INTRODUCTION

The diffusion of a dilute solute in oscillating flows has application in a wide variety of physical processes ranging from oxygen transport in respiratory processes [3] to enhanced mass transport in oscillatory liquid membranes [4]. To date, the study of diffusion in oscillatory flows has been confined to generalizations of Taylor dispersion processes [5], beginning with Aris [6] who derived the effective dispersivity of a dilute solute in viscous pulsatile flow (the limit $Sc = \nu/D \gg 1$, where ν is the kinematic viscosity and D is the molecular diffusivity) through a circular conduit. Watson [2] obtained a general expression for sinusoidal periodic flow through a conduit of arbitrary cross-section for systems with arbitrary Schmidt numbers and oscillation frequencies. Horn and Kipp [7] examined the analogous problem for oscillatory planar Couette flow. Dill and Brenner [8] have employed generalized Taylor dispersion theory [9] to examine mass transport in oscillatory flow through spatially periodic porous media.

In all of these studies only the long-time limit of the dispersion coefficient was obtained, i.e. the dispersivity which would be observed for very long tubes. For this limit it is possible to average the concentration distribution across the conduit (or across a cell in a spatially periodic structure) and obtain a series of equations for the moments of the distribution. The actual concentration distributions are never determined, and the moments are derived only for bounded flows. In the next section we derive a self-similar solution for the concentration distribution arising from an initial point distribution in an unbounded oscillating simple shear flow. The result is generalized for arbitrary time dependent simple shear flows and is shown to agree with the well-known result for a steady unbounded shear flow (see Foister and van de Ven [1]). A formal solution for diffusion from an arbitrary time dependent source distribution is also obtained. In the third section the moments of the concentration distribution in the direction of flow are calculated and the time dependent effective dispersivity is determined. In the final section the solution is applied to bounded flows through conduits of uniform cross-section, where it is shown to agree with the more general results of Aris [6] and Watson [2] in the appropriate limit.

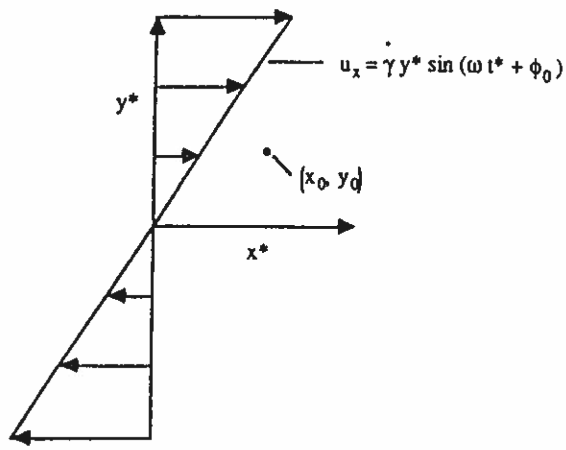

Fig. 1. The flow geometry. The flow is a periodic simple shear flow centered at the origin. The concentration distribution is initially a delta function centered at (x_0, y_0).

2. DERIVATION

Consider an initial point distribution $I\delta(x^*-x_0)\delta(y^*-y_0)$ at time $t^* = t_0$ in the oscillating infinite simple shear flow depicted in Fig. 1. We shall take the velocity to be given by $u_x = \dot{\gamma}y^* \sin(\omega t^* + \phi_0)$, where $\dot{\gamma}$ is the shear rate, ω is the angular frequency, and ϕ_0 is the initial phase shift. We restrict ourselves to diffusion in two dimensions. Since there is no motion in the z-direction, however, diffusion in this direction will be unaffected by the flow and the full three dimensional result may be recovered by simply multiplying the two-dimensional solution with the solution for one-dimensional diffusion from a point in the z-direction in the absence of convection. The concentration distribution in the dilute limit is governed by the convection–diffusion equation:

$$\frac{\partial c^*}{\partial t^*} + \dot{\gamma}y^* \sin(\omega t^* + \phi_0)\frac{\partial c^*}{\partial x^*} = D\left(\frac{\partial^2 c^*}{\partial x^{*2}} + \frac{\partial^2 c^*}{\partial y^{*2}}\right) \tag{2.1}$$

with initial and boundary conditions:

$$c^*(x, y, t_0) = I\delta(y^*-y_0)\delta(y^*-y_0); \quad c^*|_{x^*, y^* \to \infty} = 0.$$

To solve equation (2.1) we first render the equation dimensionless. We define the dimensionless variables:

$$x = \frac{x^*-x_0}{(D/\omega)^{1/2}}; \quad y = \frac{y^*-y_0}{(D/\omega)^{1/2}}; \quad t = \omega(t^*-t_0); \quad c = \frac{c^*D}{\omega I},$$

which result in the dimensionless equation:

$$\frac{\partial c}{\partial t} + (\alpha y + \beta)\sin(t+\phi)\frac{\partial c}{\partial x} = \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \tag{2.2}$$

and:

$$c(x, y, 0) = \delta(x)\delta(y); \quad c|_{x, y \to \infty} = 0,$$

where the dimensionless problem is a function only of the shear rate/angular frequency ratio $\alpha = \dot{\gamma}/\omega$, the amplitude of the periodic velocity at the point where the solute is introduced $\beta = \dot{\gamma}y_0/(D\omega)^{1/2}$ and the initial phase at $t^* = t_0$ of $\phi = \phi_0 + \omega t_0$. Note that α is equal to the amplitude of the fluid strain during the oscillatory motion.

Since there is no motion in the y-direction, diffusion in that direction will be unaffected by convection. We may therefore factor out the contribution from the purely molecular diffusion in the y-direction by defining:

$$c(x, y, t) = f(x, y, t)g(y, t) \tag{2.3}$$

where $g(y, t)$ satisfies the one-dimensional diffusion equation:

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial y^2}; \quad g(y, 0) = \delta(y) \tag{2.4}$$

which has the well-known solution:

$$g(y, t) = \frac{1}{\sqrt{4\pi t}} e^{-y^2/4t}. \tag{2.5}$$

Substituting equations (2.3) and (2.5) into the differential equation (2.2), we obtain:

$$\frac{\partial f}{\partial t} + (\alpha y + \beta)\sin(t + \phi)\frac{\partial f}{\partial x} + \frac{y}{t}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \tag{2.6}$$

with the initial condition $f(x, y, 0) = \delta(x)$. To solve this equation, we shall seek a self-similar solution such that f is not a function of x, y and t independently, but rather is a function of η and t alone, where the similarity variable η is a simple function of x, y and t. To this end we define the dimensionless variable η in the x-direction by analogy to the solution for diffusion in a steady infinite shear flow:

$$\eta = x - \beta\int_0^t \sin(t' + \phi)\, dt' - \alpha y h(t), \tag{2.7}$$

where the function $h(t)$ remains to be determined.

In defining η in this manner we are, in effect, shifting the x-coordinate by some amount that takes into account the displacement of a fluid element by the shear flow. For any given y and t there will be some value of x at which the concentration will be a maximum, and we wish to shift the x-coordinate so that $\eta = 0$ at this point. For $y = 0$ (the point in the y-direction at which the concentration was initially introduced) the local maximum in the x-direction will simply be displaced back and forth with the velocity of the shear flow at that point. This displacement is accounted for by the second term in equation (2.7). For non-zero values of y, however, the situation is more complex. In diffusing to a particular value of y the solute will have executed a random walk across streamlines with different velocities, thus in addition to the uniform displacement corresponding to the initial position, the shear flow will induce a displacement proportional to y and the strain amplitude α, as well as an unknown function of time $h(t)$, characteristic of the random walk. Substituting the definition of η into equation (2.6) we obtain the new equation:

$$\left(\frac{\partial f}{\partial t}\right)_\eta - \alpha y\left[h' + \frac{h}{t} - \sin(t + \phi)\right]\left(\frac{\partial f}{\partial \eta}\right)_t = (1 + \alpha^2 h^2)\left(\frac{\partial^2 f}{\partial \eta^2}\right)_t \tag{2.8}$$

with the initial condition $f(\eta, 0) = \delta(\eta)$.

In order for the self-similar solution to exist, we require:

$$h' + \frac{h}{t} - \sin(t+\phi) = 0 \tag{2.9}$$

so that the y-dependent term in equation (2.8) will vanish. We also require $h(t)$ to satisfy the initial condition $h(0) = 0$, since the displacement of η from x should vanish at $t = 0$. The first order ODE has the simple solution:

$$h(t) = \frac{1}{t}\int_0^t t' \sin(t'+\phi)\,\mathrm{d}t' = \left(\frac{\sin t}{t} - \cos t\right)\cos\phi + \left(\sin t + \frac{\cos t - 1}{t}\right)\sin\phi \tag{2.10}$$

and yields the transformed PDE for $f(\eta, t)$:

$$\frac{\partial f}{\partial t} = (1+\alpha^2 h^2)\frac{\partial^2 f}{\partial \eta^2}, \tag{2.11}$$

where $h(t)$ is as defined above.

We may further simplify the differential equation for $f(\eta, t)$ if we define the new dimensionless time variable ξ:

$$\xi = \int_0^t (1+\alpha^2 h^2)\,\mathrm{d}t' \tag{2.12}$$

which may be explicitly evaluated to give:

$$\xi = t + \alpha^2\left\{\frac{1}{2}t + \frac{1}{2}\sin(t+\phi)\cos(t+\phi) - \frac{1}{t}[\sin(t+\phi) - \sin\phi]^2\right\}. \tag{2.13}$$

Equation (2.11) thus becomes

$$\frac{\partial f}{\partial \xi} = \frac{\partial^2 f}{\partial \eta^2}; \quad f(\eta, 0) = \delta(\eta), \tag{2.14}$$

which has the very simple solution:

$$f(\eta, \xi) = \frac{1}{\sqrt{4\pi\xi}}\,\mathrm{e}^{-\eta^2/4\xi} \tag{2.15}$$

and hence the overall concentration distribution in dimensional variables is given by:

$$c^*(x^*, y^*, t^*) = \frac{I}{(D/\omega)}\frac{1}{4\pi\sqrt{\xi t}}\,\mathrm{e}^{-y^2/4t}\,\mathrm{e}^{-\eta^2/4\xi}, \tag{2.16}$$

where ξ, η, y and t are as defined above. Plots of the concentration distribution for selected values of α, β, ϕ and t are given in Fig. 2. Note that contours of constant concentration are simply ellipses whose inclinations relative to the y-axis are periodic in time.

It is interesting to note that while the concentration distribution given by equation (2.16) was derived for purely sinusoidal time variations in the amplitude of a shear flow, the result can be easily generalized for any time dependent simple shear $u_x = \dot{\gamma} y^* s(t^* - t_0)$, where $s(t)$ is at least piecewise continuous. In this case, we define:

$$\eta = x - \beta\int_0^t s(t')\,\mathrm{d}t' - \alpha y h(t), \tag{2.17}$$

where $h(t)$ is the integral:

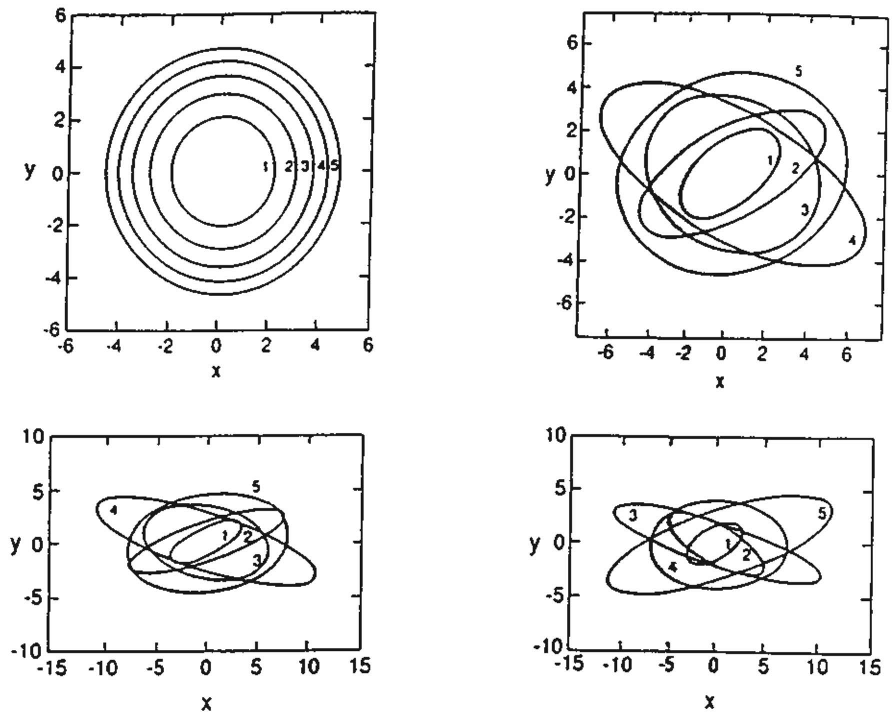

Fig. 2. Concentration profiles. The concentration contours given by $c(x, y, t)/c(0, 0, t) = 0.5$ are plotted for $\beta = 0$ and the dimensionless times $t = n\pi/2$, where $n = 1, 2, 3, 4$ and 5. The contours are given for: a: $\alpha = 0$ (no shear); b: $\alpha = 1$, $\phi = 0$; c: $\alpha = 2$, $\phi = 0$; d: $\alpha = 2$, $\phi = \pi/2$.

$$h(t) = \frac{1}{t}\int_0^t t' s(t')\,\mathrm{d}t' \tag{2.18}$$

and t, y and x are defined by analogy with the result for purely sinusoidal variations. The concentration distribution given by (2.16) in terms of these new variables is unchanged. This result may also be obtained by a generalization of the approach followed by Foister and van de Ven [1] to time dependent flows. For the particular case of a steady infinite shear flow we have $s(t) = 1$, thus $h(t) = t/2$, $\eta = x - \beta t - (\alpha/2)yt$, and $\xi = t + (\alpha^2/12)t^3$, yielding the well-known self-similar solution for steady flow:

$$c^*(x^*, y^*, t^*) = \frac{I}{4\pi D(t^* - t_0)} \frac{\exp\left\{\dfrac{-(y^* - y_0)^2}{4D(t^* - t_0)} + \dfrac{-[x^* - \frac{1}{2}\dot{\gamma}(t^* - t_0)(y^* + y_0)]^2}{4D(t^* - t_0)[1 + \frac{1}{12}\dot{\gamma}^2(t^* - t_0)^2]}\right\}}{\sqrt{1 + \frac{1}{12}\dot{\gamma}^2(t^* - t_0)^2}}. \tag{2.19}$$

Equation (2.16) may also be generalized for an arbitrary time dependent distributed source $I(x_0, y_0, t_0)$:

$$c(x^*, y^*, t^*) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{I(x_0, y_0, t_0)}{(D/\omega)} \frac{1}{4\pi\sqrt{\xi t}} \mathrm{e}^{-y^2/4t}\,\mathrm{e}^{-\eta^2/4\xi}\,\mathrm{d}x_0\,\mathrm{d}y_0\,\mathrm{d}t_0, \tag{2.20}$$

where the dimensionless variables in equation (2.20) are defined as before.

3. MOMENTS OF THE DISTRIBUTION

In order to more easily visualize the effect of oscillatory shear flow on mass transport it is useful to calculate the moments of the concentration distribution in the direction of flow. These moments may be determined quite easily from the concentration distribution given by equation (2.16). The dimensionless mean position of the concentration profile (first moment) is defined by:

$$\bar{x} \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xc(x, y, t)\, \mathrm{d}x\, \mathrm{d}y, \tag{3.1}$$

which may be evaluated to yield:

$$\bar{x} \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\eta + \beta \int_0^t \sin(t' + \phi)\, \mathrm{d}t' + \alpha y h(t)\right) \frac{1}{4\pi\sqrt{\xi t}} \mathrm{e}^{-\eta^2/4\xi} \mathrm{e}^{-y^2/4t}\, \mathrm{d}\eta\, \mathrm{d}y \tag{3.2}$$

$$= \beta \int_0^t \sin(t' + \phi)\, \mathrm{d}t'$$

and from which it is seen that the mean position of the concentration profile simply moves back and forth with the periodic motion of the shear flow evaluated at y_0. Of more interest is the variance (second moment) of the distribution, defined by:

$$\sigma_x^2 \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{x})^2 c(x, y, t)\, \mathrm{d}x\, \mathrm{d}y \tag{3.3}$$

and which is evaluated to yield:

$$\sigma_x^2 \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\eta + \alpha y h(t)]^2 \frac{1}{4\pi\sqrt{\xi t}} \mathrm{e}^{-\eta^2/4\xi} \mathrm{e}^{-y^2/4t}\, \mathrm{d}\eta\, \mathrm{d}y \tag{3.4}$$

$$= \int_{-\infty}^{\infty} (2\xi + \alpha^2 h^2 y^2) \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-y^2/4t}\, \mathrm{d}y = 2\xi + 2t\alpha^2 h^2$$

or, substituting in the definition of ξ:

$$\sigma_x^2 = 2t\left[1 + \alpha^2 h^2 + \frac{\alpha^2}{t} \int_0^t h^2(t')\, \mathrm{d}t'\right] \tag{3.5}$$

and in terms of dimensional quantities:

$$\sigma_x^{*2} = 2D(t^* - t_0)\left[1 + \alpha^2 h^2 + \frac{\alpha^2}{(t^* - t_0)} \int_0^{t^* - t_0} h^2(t')\, \mathrm{d}t'\right]. \tag{3.6}$$

For periodic flow the variance in equation (3.5) has the explicit form:

$$\sigma_x^2 = 2t\{1 + \alpha^2[\tfrac{1}{2} + \cos^2(t + \phi)] + \frac{\alpha^2}{t}[2\cos(t + \phi)\sin\phi - \tfrac{3}{2}\sin(t + \phi)\cos(t + \phi)]\}, \tag{3.7}$$

where the last term in equation (3.7) becomes negligible at large times. A plot of the variance as a function of time is given in Fig. 3.

From equations (3.5) and (3.7) it may be seen that the growth of the variance of the concentration profile in the x-direction is the sum of contributions from three terms. The first term in (3.5) simply represents molecular diffusion in the x-direction in the absence of

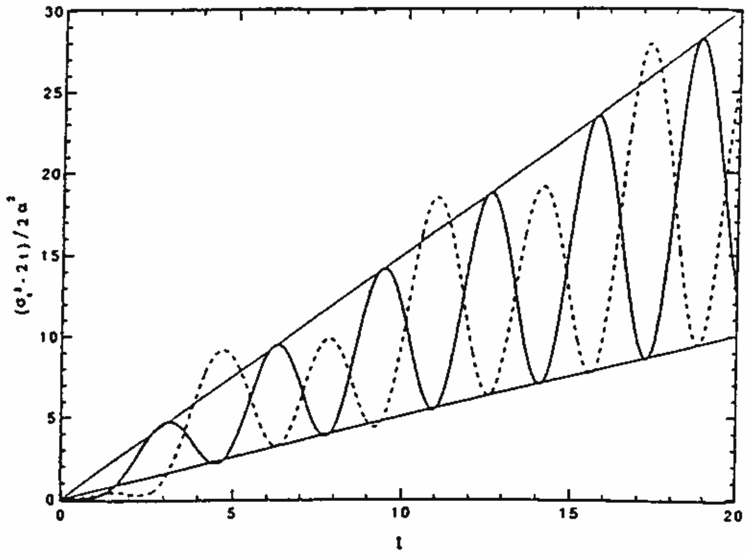

Fig. 3. Variance of the concentration profile. The dimensionless variance in the x-direction due to convection (the contribution due to molecular diffusion in the x-direction has been subtracted off) reduced by the amplitude of the oscillatory strain is plotted vs time. The solid curve is for $\phi = 0$ and the dashed curve for $\phi = \pi/2$. The upper and lower lines are the limiting asymptotes at large time of $3t/2$ and $t/2$, respectively.

the imposed flow field. The second term arises from the transient deformation in the x-direction by the shear flow of a concentration profile which extends in the y-direction. The amplitude of the periodic strain imposed on the shear flow is constant, thus as the solute diffuses further from its origin in the y-direction it will be spread out in the x-direction over a greater distance, resulting in an increase in the second moment. For the periodic flow considered here, the strain the fluid undergoes is periodic and thus the contribution of the second term to the variance is also periodic—although its amplitude grows linearly in time (i.e. the square of the diffusion length in the y-direction multiplied by the square of the strain amplitude), it does not lead to a permanent increase in the second moment in the same manner as does molecular diffusion in the x-direction.

The third term in equation (3.5) is the increase in dispersion arising from the cooperative effects of diffusion in the y-direction and convection in the x-direction. A physical understanding of this term may be gained with the aid of Fig. 4. Consider a solute molecule migrating from point A to point B in a shear flow. The molecule may reach point B either by pure molecular diffusion in the x-direction, or by diffusing in the y-direction to a faster moving streamline, being convected in the x-direction, and then diffusing back to B. While other random walk paths are just as likely, the combination of convection and diffusion will lead to an increase in the dispersion of the solute in the direction of flow. Note that while the degree of dispersion has a periodic part for short times, at long times it approaches the linear increase in variance with the time expected for a diffusion process. For large values of α, it is this convective–diffusion term which dominates the dispersion of the solute in the x-direction.

4. APPLICATION TO BOUNDED FLOWS

While the enhancement in mass transfer given above for an oscillating shear flow has been derived for an unbounded flow, it is in bounded systems where the effect is practically

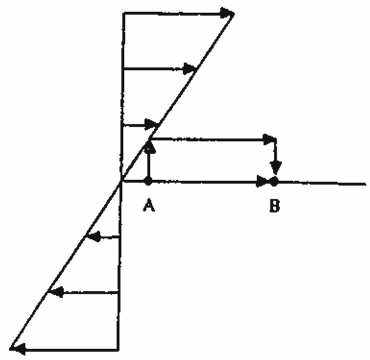

Fig. 4. Influence of shear on migration. A solute molecule may migrate from point A to point B by either (1) molecular diffusion in the x-direction or (2) diffusion in the y-direction followed by convection in the x-direction and diffusion in the y-direction. The additional migration path made possible by convection increases the rate of dispersion in the direction of flow.

realizable. In order for equation (2.16) to apply to a bounded system, we require that the flow must locally appear to the solute molecules to be an oscillating infinite shear flow on the timescale of the period of flow oscillation. Consider the oscillatory axial flow through the conduit depicted in Fig. 5. The cross-section may be arbitrary, however we require it to be unchanging along the length L of the conduit. If the cross-section of the conduit is characterized by some length scale a, we may achieve an oscillatory shear flow provided that $\omega a^2/\nu \ll 1$, where ν is the kinematic viscosity of the fluid. This condition is equivalent to requiring that the fluid is sufficiently viscous that the shear layer produced by the walls propagates a distance much greater than the radius of the conduit during the period of oscillation. The flow will appear to approximate an infinite shear flow to a solute molecule on an individual streamline if we impose the additional condition that $\omega a^2/D \gg 1$, i.e. that during one oscillation the molecule diffuses a length much smaller than the radius of the conduit (the length of gradients in the local shear rate). Note that these conditions require that the Schmidt number $Sc = \nu/D$ of the system be very large.

Under these conditions, the effective dispersion coefficient K in the axial direction along any streamline at long times may be determined from equation (3.7):

$$K = D\left(1+\frac{\alpha^2}{2}\right), \tag{4.1}$$

where the periodic contribution to the dispersion has been neglected. Note that this contribution to the variance arose solely from an unbounded molecular diffusion in the y-direction being deformed by the periodic shear flow. This effect will be absent in a bounded flow since diffusion in the y-direction is ultimately limited by the presence of the walls of

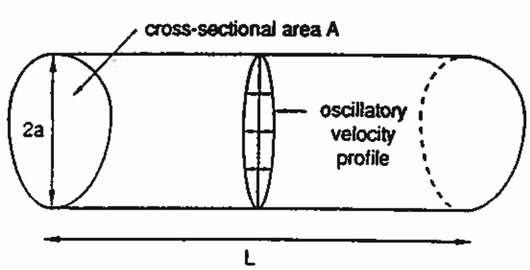

Fig. 5. Conduit with uniform cross-section. Fluid undergoes oscillatory flow in a conduit of length L having a uniform cross-section with diameter $2a$ and area A.

the conduit, and thus this contribution to the dispersion in the x-direction will similarly be bounded and will not grow in amplitude with time as is characteristic of a diffusion process.

For a long conduit at steady-state ($t^*D/a^2 \gg 1$) there will be no time-average concentration gradients across streamlines. As a consequence, the average dispersivity experienced by the solute in steady transport down the conduit is simply the area average of the effective dispersivity along each streamline. If the conduit area is given by A, then the average dispersivity $\bar{K}$ is given by:

$$\bar{K} = D\left(1 + \frac{1}{A}\int_A \frac{1}{2}\frac{\dot{\gamma}^2}{\omega^2}\,\mathrm{d}A\right). \tag{4.2}$$

We may also relate the average dispersivity to the total rate of viscous work done in the conduit. If the conduit is of length L then the time average rate of viscous work is given by:

$$\dot{W} = \mu L \int_A \tfrac{1}{2}\dot{\gamma}^2\,\mathrm{d}A, \tag{4.3}$$

wher· μ is the fluid viscosity. Combining equations (4.2) and (4.3) we obtain:

$$\bar{K} = D\left(1 + \frac{\dot{W}}{\mu L A \omega^2}\right), \tag{4.4}$$

which provides a convenient way of experimentally estimating the dispersivity of a solute at high oscillation frequencies (but infinite v/D) in conduits with complex cross-sections. For the particular case of a circular tube of radius a we may use Poiseuille's Law to relate the viscous dissipation to the amplitude of the tidal displacement:

$$\bar{K} = D\left[1 + \left(\frac{\Delta x}{a}\right)^2\right], \tag{4.5}$$

which is in agreement with the result given by Aris [6] and Watson [2] in the correct limits. The high frequency limit of the enhancement in dispersivity calculated here is the maximum enhancement which can be achieved for a given tidal displacement at large Schmidt numbers. It must be emphasized that the results above for bounded flows are valid only for viscous flow in long conduits of uniform cross-section at high frequencies. Low frequencies, end effects, or varying cross-sections will greatly complicate the analysis.

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NOMENCLATURE

a	conduit radius
A	cross-sectional area of conduit
c	concentration
c^*	dimensionless concentration
D	molecular diffusivity
f	concentration distribution in x-direction
g	concentration distribution in y-direction
h	time varying strain amplitude
I	total amount of solute
K	effective dispersivity along a streamline
$\bar{K}$	average effective dispersivity

L	length of conduit
s	time varying shear amplitude
t	dimensionless time
t^*	time
t_0	initial time
u_x	fluid velocity
$\dot{W}$	rate of viscous work
x	dimensionless x-coordinate
$\bar{x}$	average x-position
x^*	x-coordinate
x_0	initial x-position
y	dimensionless y-coordinate
y^*	y-coordinate
y_0	initial y-position
z	dimensionless z-coordinate

Greek symbols

α	amplitude of oscillatory strain
β	dimensionless velocity at $y^* = y_0$
$\dot{\gamma}$	amplitude of shear rate
δ	Dirac delta function
Δx	amplitude of tidal displacement
η	transformed x-coordinate
μ	fluid viscosity
ν	fluid kinematic viscosity
ξ	transformed time
π	constant
σ_x^2	dimensionless variance in x-direction
σ_x^{*2}	dimensional variance in x-direction
ϕ	phase at $t = 0$
ϕ_0	phase at $t^* = 0$
ω	angular frequency

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