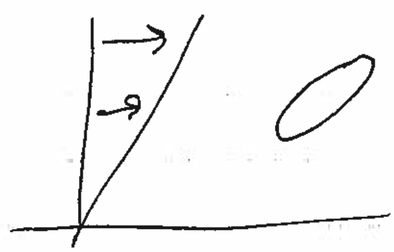
More complex inspectional analysis:
Trial functions

Dispersion-Induced spread of a dilute emulsion in Couette flow.

Consider a droplet in a shear flow. The droplet causes it to deform:

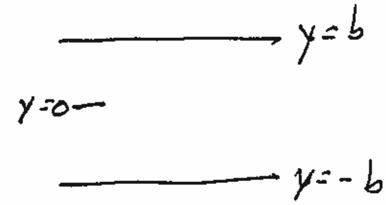


and the deformation causes it to drift away from the wall(s) of the device.

Droplet drift is imp.: influences cell dist near walls in blood flow, mass transfer in emulsions, etc.

In addition, as droplets tumble over one another, they experience a diffusivity - concentration gradients will smear out.

Diffusivity scales as:



$$D \sim \lambda \phi \dot{\gamma} a^2$$

$\uparrow$ may be a function of Ca, but theory suggests it is weak.

The droplet drift velocity is very complex, but near the center of a Couette gap it is well approximated by:

$$u \cong -A \, Ca \, \frac{\dot{\gamma} a^3}{b^2} \, \frac{y}{b}$$

where $Ca = \frac{\dot{\gamma} \mu_o a}{\sigma} \equiv$ capillary #

$A = f^{\underline{n}}$ of viscosity ratio

At S.S. a drop will drift to the centerline $y=0$. Diffusion will cause it to drift away. We want to use this process to measure λ - dimensionless dispersion coef - for the first time.

$$\frac{\partial \phi}{\partial t} + \frac{\partial (u\phi)}{\partial y} = \frac{\partial}{\partial y}\left(D \frac{\partial \phi}{\partial y} \right)$$

e.g., $N_y = u\phi - D\frac{\partial \phi}{\partial y}$, $\frac{\partial \phi}{\partial t} = -\frac{\partial N_y}{\partial y}$

Thus:

$$\frac{\partial \phi}{\partial t} - A\,Ca\,\frac{\dot{\gamma} a^3}{b^2}\frac{\partial}{\partial y}\left(\phi \frac{y}{b}\right) = \lambda \dot{\gamma} a^2 \frac{\partial}{\partial y}\left(\phi \frac{\partial \phi}{\partial y}\right)$$

Let's render this dimensionless. We let:

$$y^* = y/L_y \quad , \quad t^* = t/t_c \quad , \quad \phi^* = \phi/\phi_c$$

Thus:

$$\frac{\phi_c}{t_c}\frac{\partial \phi}{\partial t^*} - A\,Ca\,\frac{\dot{\gamma} a^3}{b^3}\phi_c \frac{\partial(\phi y^*)}{\partial y^*} = \phi_c^2 \frac{\lambda \dot{\gamma} a^2}{L_y^2}\frac{\partial}{\partial y^*}\left(\phi \frac{\partial \phi}{\partial y^*}\right)$$

Multiplying by t_c/ϕ_c:

$$\frac{\partial \phi^*}{\partial t^*} - \left[A\,Ca\,\frac{\dot{\gamma} a^3}{b^3} t_c\right]\frac{\partial(\phi^* y^*)}{\partial y^*} = \left[\frac{\lambda \dot{\gamma} a^2 t_c \phi_c}{L_y^2}\right]\frac{\partial}{\partial y^*}\left(\phi^* \frac{\partial \phi^*}{\partial y^*}\right)$$

~~choose t_c, L_y s.t. quantities in brackets are equal~~

We also have the B.C.'s:

$$\left.\frac{\partial \phi}{\partial y}\right|_{y=0} = 0 \quad \text{(symmetry condition)}$$

and

$$\left.\phi\right|_{y=y_e} = 0 \quad \text{(edge of emulsion)}$$

and

$$\int_0^{y_e} \phi \, dy = \bar{\phi}_0 b$$

conservation condition.

We won't worry about I.C. yet

Let's render conservation condition dimensionless:

$$\int_0^{y_e^*} \phi^* dy^* = \left[\frac{\bar{\phi}\, b}{\phi_c L_y}\right]$$

we have the three dimensionless groups:

$$\left[A\, Ca \frac{\dot{\gamma} a^3}{b^3} t_c\right], \left[\frac{\lambda \dot{\gamma} a^2 t_c \phi_c}{L_y^2}\right], \left[\frac{\bar{\phi} b}{\phi_c L_y}\right]$$

We choose L_y, t_c, ϕ_c s.t. all are unity!

Thus we get:

$$t_c = \left\{ A\,Ca\,\frac{\dot{\gamma} a^3}{b^3} \right\}^{-1}$$

which is easy. To get the other two we have:

$$\frac{\lambda \dot{\gamma} a^2 t_c \phi_c}{L_y^2} = 1 \;, \quad \frac{\bar{\phi} b}{\phi_c L_y} = 1$$

Thus $L_y = \dfrac{\bar{\phi} b}{\phi_c}$

and:

$$\frac{\lambda \dot{\gamma} a^2 t_c \phi_c}{\left(\dfrac{\bar{\phi} b}{\phi_c}\right)^2} = 1$$

$$\phi_c = \left(\frac{\bar{\phi}^2 b^2}{\lambda \dot{\gamma} a^2 t_c} \right)^{1/3}$$

$$= \left(\frac{\bar{\phi}^2 \cancel{b^2} A\,Ca \frac{\cancel{\dot{\gamma}} a^{\cancel{3}}}{b^{\cancel{3}}}}{\lambda \cancel{\dot{\gamma}} \cancel{a^2}} \right)^{1/3} = \left(\frac{\bar{\phi}^2 A\,Ca}{\lambda} \frac{a}{b} \right)^{1/3}$$

and, finally, $L_y = \dfrac{\bar{\phi} b}{\phi_c} = b \left(\dfrac{\lambda \bar{\phi}}{A\,Ca} \dfrac{b}{a} \right)^{1/3}$

Note that for our approximation of the drift velocity to be valid, we require

$$L_y << b, \text{ or alt. } \left(\frac{\lambda \bar{\phi}}{A\,Ca}\,\frac{b}{a}\right)^{1/3} << 1$$

otherwise the emulsion will get too close to the walls and nonlinearities (in y) will come into play!

specifically, we require $y_e^* << \frac{b}{L_y}$

where y_e^* is the edge of the emulsion.

Ok, we have the P.D.E:

$$\frac{\partial \phi^*}{\partial t^*} - \frac{\partial}{\partial y^*}\left(\phi^* y^*\right) = \frac{\partial}{\partial y^*}\left(\phi^* \frac{\partial \phi^*}{\partial y^*}\right)$$

with B.C. $\left.\frac{\partial \phi^*}{\partial y^*}\right|_{y^*=0} = 0 \qquad \left.\phi^*\right|_{y^*=y_e^*} = 0$

$$\int_0^{y_e^*} \phi^* \, dy^* = 1$$

Let's solve for the steady-state:

$\phi_\infty^* = f^n(y^*)$ alone (e.g. $\frac{\partial \phi_\infty^*}{\partial t^*} = 0$)

Thus $$-\frac{d}{dy^*}(\phi_\infty^* y^*) = \frac{d}{dy^*}\left(\phi_\infty^* \frac{d\phi_\infty^*}{dy^*}\right)$$

Integrating:

$$-\phi_\infty^* y^* = \phi_\infty^* \frac{d\phi_\infty^*}{dy^*} + C$$

Apply B.C. at $y^* = 0$: $\therefore\ C = 0$

Divide out by ϕ_∞^*:

$$\frac{d\phi_\infty^*}{dy^*} = -y^*$$

$$\phi_\infty^* = -\frac{1}{2} y^{*2} + C$$

$$\phi_\infty^*\Big|_{y^* = y_e^*} = 0 \quad \therefore \quad C = \frac{1}{2} y_e^{*2}$$

So $\phi_\infty^* = \frac{1}{2}\left(y_e^{*2} - y^{*2}\right)$ - a parabola!

Now we apply the conservation eq'n to get y_e^*:

$$\int_0^{y_e^*} \phi_\infty^* \, dy^* = 1 = \frac{1}{2}\left(y_e^{*3} - \frac{1}{3} y_e^{*3} \right) = \frac{1}{3} y_e^{*3}$$

$$\text{Thus} \quad y_e^{*3} = 3 \qquad y_e^* = (3)^{1/3} = 1.44$$

Note that for the emulsion to fit in the gap, we require

$$1.44 = y_e^* < \frac{b}{L_y} \,.$$

or, alternatively,

$$\frac{\lambda \bar{\phi}}{A\, Ca} \frac{b}{a} < \frac{1}{3}$$

This dimensionless group should be much less than 1/3 for the equation to be asymptotically valid.

Now we do the interesting part. We shall seek to solve the transcent problem for the evolution of the concentration distrib. from one sheer rate to another.

If the solution is self-similar, the <u>shape</u> of the concentration profile will be the same over time, but its magnitude and width will change.
(e.g., it starts as a parabola & finishes with that shape - maybe it keeps it during the transition too!)

Due to the existence of a steady-state profile, a simple affine stretching will <u>not</u> result in a similarity solution. Instead we seek a trial function solution:

$$\phi^* = h(t^*)\, f(\eta) \; ; \quad \eta = \frac{y^*}{s(t^*)}$$

Note that we put all the complexity in t^* - the variable which appears as the lowest deriv. in the equation. We now develop equations for $s(t^*)$ and $h(t^*)$ which must be satisfied for the problem to be self-similar.

Thus:

$$\frac{\partial \phi^*}{\partial y^*} = \frac{h}{s} f' \qquad \phi^* \frac{\partial \phi^*}{\partial y^*} = \frac{h^2}{s} f f'$$

$$\frac{\partial}{\partial y^*}\left(\phi^* \frac{\partial \phi^*}{\partial y^*}\right) = \frac{h^2}{s^2}\,(f f')'$$

$$y^* \phi^* = y^* h f = s h \eta f$$

$$\frac{\partial}{\partial y^*}(y^* \phi^*) = \frac{sh}{s}(\eta f)' = h(\eta f)'$$

$$\frac{\partial \phi^*}{\partial t^*} = \frac{\partial}{\partial t^*}\left(h f(\eta)\right) = h' f + h f' \frac{\partial \eta}{\partial t}$$

$$= h' f - h \frac{s'}{s} \eta f'$$

Thus:

$$h' f - h \frac{s'}{s} \eta f' - h(\eta f)' = \frac{h^3}{s^2}\,(f f')'$$

with B.C.'s $\quad f'(0) = 0$

$$f(\eta_e) = 0$$

$$\int_0^{\eta_e} h s f \, d\eta = 1$$

In order for the problem to be self-similar, h & s must be chosen so that it is not a function of t^* & η, but rather η alone.

from the integral condition we clearly require that

$$\underline{hs = cst}$$

We choose this constant $= 1$ (it doesn't make any difference)

Thus $h = \frac{1}{s}$

and $h' = \frac{dh}{dt^*} = \frac{d}{dt^*}\left(\frac{1}{s}\right) = -\frac{s'}{s^2}$

Substituting in:

$$-\frac{s'}{s^2} f - \frac{s'}{s^2} \eta f' - \frac{1}{s}(\eta f)' = \frac{1}{s^4}(ff')'$$

$$\equiv -\frac{s'}{s^2}(\eta f)' - \frac{1}{s}(\eta f)' = \frac{1}{s^4}(ff')'$$

$$-(s^2 s' + s^3)(\eta f)' = (ff')'$$

The problem will be self-similar if $s(t^*)$ satisfies the differential equation:

$$s^2 s' + s^3 = \underline{cst} = 1$$

where again we take the constant to be unity for simplicity!

We may solve this easily if we do the variable substitution:

$$p = s^3 \qquad \therefore \; p' = 3s^2 s'$$

and hence:

$$\frac{1}{3} p' + p = 1$$

We need to choose some B.C. for p to obtain a complete solution, ~~Recalling that~~ however p will just be some exponential decay to an asymptotic value of unity. We'll get back to this later.

Now for the ODE:

$$(ff')' = -(\zeta f)'$$

$$f'(0) = 0 \qquad f(\zeta_e) = 0$$

$$\int_0^{\zeta_e} f \, d\zeta = 1$$

This is the same problem we had at steady-state! The solution is just:

$$f = \frac{1}{2}\zeta_e^2\left(1 - \frac{\zeta^2}{\zeta_e^2}\right) \quad ; \quad \zeta_e = (3)^{1/3}$$

Now we look at the I.C. for s (or ρ). We are looking at the evolution of the concentration profile from one steady-state at one shear rate to another at a different shear rate. From the solution to the transient equation it will just be a parabola at all times! Only its width & magnitude will change. Thus we take $s(0) = s_0$ s.t. if the initial edge of the suspension is at y_0^*, then:

$$\frac{y_0^*}{s_0} = (3)^{1/3}$$

$$s_0 = \frac{y_0^*}{\sqrt[3]{3}} \quad , \quad \rho_0 \equiv s_0^3 = \frac{y_0^{*3}}{3}$$

and hence:

$$\rho = 1 + (\rho_0 - 1)\, e^{-3t^*}$$

Thus we get the full solution:

$$\phi^* = \frac{1}{s} f(\eta), \quad \eta = \frac{y^*}{s(t^*)}$$

$$s = \left\{ 1 + \left(\frac{y_o^{*3}}{3} - 1 \right) e^{-3t^*} \right\}^{1/3}$$

$$f = \frac{1}{2} \eta_e^2 \left(1 - \frac{\eta^2}{\eta_e^2} \right) \qquad \eta_e = (3)^{1/3}$$

We can then compare this solution to experimental observations of concentration profiles to get the dimensionless dispersivity λ, which can be used to study emulsion flows in other geometries.

Note that any self-similar solution revealed by affine stretching will also be revealed by trial functions, however the trial function approach is alot messier.

Self similar soln for drop migration in tube flow in the dilute limit

The governing equation is

$$\frac{\partial \phi}{\partial t} = -\frac{1}{r}\frac{\partial (N_r r)}{\partial r}$$

$$N_r = -k_d \, Ca \, 4\frac{a^2}{R^2}\frac{r}{R}\phi - K_\sigma \frac{\phi^2}{\sigma}\frac{d\sigma}{dr}\dot{\gamma} a^2 - \lambda \phi \dot{\gamma} a^2 \frac{\partial \phi}{\partial r}$$

For tube flow, $\frac{d\sigma}{dr} = \frac{\sigma}{r}$

and

$$\dot{\gamma} = \frac{4U r}{R^2}$$

$$N_r = -k_d \, Ca \frac{U a^2}{R}\frac{r}{R}\phi - K_\sigma \phi^2 \, 4U\frac{a^2}{R^2} - \lambda \, 4U r \frac{a^2}{R^2}\phi \frac{\partial \phi}{\partial r}$$

$$2\int_0^{\frac{r_e}{R}} \phi\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) = \phi_0 \qquad \& \qquad \phi\Big|_{r^e = r_e} = 0$$

where r_e is the radial position demarcating the clear fluid region

To render the variables dimensionless, we use

$$r^* = \frac{r}{R}, \qquad t^* = \frac{t}{\frac{R^2}{\left(\frac{4 U \lambda a^2}{R^2}\phi_0\right)}}$$

$$\phi^* = \frac{\phi}{\phi_0}$$

$$\frac{\partial \phi^*}{\partial t^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left[r^*\left(\alpha\, r^* \phi^* + \beta\, \phi^{*2} + r^* \phi^* \frac{d\phi^*}{dr^*}\right)\right]$$

$$\int_0^{r_e^*} \phi^* r^* \, dr^* = \frac{1}{2}$$

where $\alpha = \dfrac{k_d \, C_a \, R}{4 \lambda \phi_0}$

$$\beta = \frac{k_\sigma}{\lambda}$$

Consider the transformation

$$\phi^* = \frac{f(\eta)}{h(t^*)} \qquad \eta = \frac{x^*}{s(t^*)}$$

Substituting these in the integral concentration condition

$$\frac{s^2(t^*)}{h(t^*)} \int_0^{\eta_e^*} f(\eta)\, \eta \, d\eta = \frac{1}{2}$$

$$\Rightarrow \quad h(t^*) = s^2(t^*)$$

$$\int_0^{\eta_e} f(\eta)\, \eta \, d\eta = \frac{1}{2}$$

$$\left.\frac{\partial}{\partial t^*}\right|_{x^*} = \left.\frac{\partial}{\partial t^*}\right|_{\eta^*} + \left.\frac{\partial}{\partial \eta}\right|_{t^*}\left(\frac{-\eta\, s'(t^*)}{s(t^*)}\right),$$

$$\Rightarrow \left.\frac{\partial \phi^*}{\partial t^*}\right|_{x^*} = \left.\frac{\partial \phi^*}{\partial t^*}\right|_{\eta^*} - \eta\frac{s'(t^*)}{s(t^*)}\left.\frac{\partial \phi^*}{\partial \eta}\right|_{t^*}$$

$$= -\frac{2f}{s^3}s' \cancel{f} - \eta\frac{s'}{s^3}f'$$

$$= -\frac{s'}{s^3}\left[\eta f' + 2f\right]$$

$$= -\frac{s'}{s^3\eta}\left[\eta^2 f' + 2\eta f\right]$$

$$= -\frac{s'}{s^3\eta}\left(\eta^2 f\right)'$$

$$\left.\frac{\partial}{\partial x^*}\right|_{t^*} = \left.\frac{\partial}{\partial \eta}\right|_{t^*}\left.\frac{\partial \eta}{\partial x^*}\right|_{t^*}$$

$$= \frac{1}{s(t)}\left.\frac{\partial}{\partial \eta}\right|_{t^*}$$

<u>Drop deformation migration term</u>

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(\alpha r^{*2}\phi^*\right) = \frac{1}{s^2\eta}\frac{\partial}{\partial\eta}\left(\alpha s^2\eta^2\frac{f}{s^2}\right)$$

$$= \frac{\alpha}{s^2\eta}\frac{\partial}{\partial\eta}\left(\eta^2 f\right)$$

$$= \frac{\alpha}{s^2\eta}\left(\eta^2 f\right)'$$

<u>Shear-stress-gradient term</u>

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(\beta r^*\phi^{*2}\right) = \frac{\beta}{s^2\eta}\frac{\partial}{\partial\eta}\left(s\eta\frac{f^2}{s^4}\right)$$

$$= \frac{\beta}{s^5\eta}\left(\eta f^2\right)'$$

<u>Conc. gradient migration term</u>

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^{*2}\phi^*\frac{\partial\phi^*}{dr^*}\right) = \frac{1}{s^2\eta}\frac{\partial}{\partial\eta}\left(s^2\eta^2\frac{f}{s^2}\frac{f'}{s^3}\right)$$

$$= \frac{1}{s^5\eta}\frac{\partial}{\partial\eta}\left(\eta^2 f f'\right)$$

$$= \frac{1}{s^5 \eta}\left(\eta^2 f f'\right)'$$

The concentration balance equation becomes

$$-\frac{s'}{s^3 \eta}\left(\eta^2 f\right)' = \frac{\alpha}{s^2 \eta}\left(\eta^2 f\right)' + \frac{\beta}{s^5 \eta}\left(\eta f^2\right)'$$

$$+ \frac{1}{s^5 \eta}\left(\eta^2 f f'\right)$$

$$\therefore \left(\beta \eta f^2 + \eta^2 f f'\right)' + \left(\eta^2 f\right)'\left[\alpha s^3 + s' s^2\right] = 0$$

To get a self-similar solution,

$$\alpha s^3 + s' s^2 = \text{constant}$$

$$\Rightarrow \quad s(t) = \left[\frac{1}{\alpha} + \left(\frac{1}{\alpha_0} - \frac{1}{\alpha}\right)\exp(-3\alpha t^*)\right]^{1/3}$$

The governing equation becomes

$$\eta^2 f f' + \beta \eta f^2 + \eta^2 f = 0$$

$$\eta f' + \beta f + \eta = 0$$

$$\eta^\beta f' + \beta \eta^{\beta-1} f + \eta^\beta = 0$$

$$(\eta^\beta f)' + \eta^\beta = 0$$

$$f = \frac{1}{\eta^\beta}\left[\frac{\eta_e^{\beta+1} - \eta^{\beta+1}}{\beta+1}\right]$$

$$\int_0^{\eta_e} f \eta \, d\eta = \frac{1}{2}$$

$$\int_0^{\eta_e} \frac{\eta^{1-\beta}\left[\eta_e^{\beta+1} - \eta^{\beta+1}\right]}{\beta+1} d\eta = \frac{1}{2}$$

$$\eta_e = \left[\frac{3}{2}(2-\beta)\right]^{1/3}$$

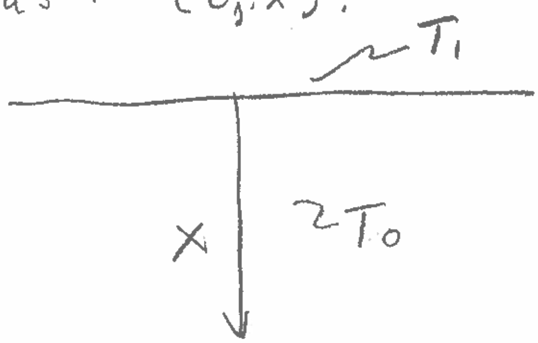
Unsteady Cond.

Up to now, have considered steady probs: in solids or in fluids w/ conv.

Now look at unsteady problem: Restrict to solids, but later look at unsteady conv. probs: instability

Consider solid, semi-inf. slab at T_0

at $t=0$ raise surf. to temp T_1. What is T as $f^n(t, x)$?



solid, so $\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t}$

$$\therefore \frac{\partial T}{\partial t} = \alpha \nabla^2 T \equiv \alpha \frac{\partial^2 T}{\partial x^2} \Rightarrow \text{one dim. cond.}$$

Apply dim. anal. : No ref. length scale!

$$\therefore T^* \equiv \frac{T - T_0}{T_1 - T_0} = f^n\left(\frac{x}{\sqrt{\alpha t}}\right) \quad \therefore \eta = \frac{x}{\sqrt{\alpha t}}$$

after time t energy diffuses distance $\sim (\alpha t)^{1/2}$, known as diffusion length substituting in:

$$\frac{\partial T^*}{\partial x} = \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x^*} = f'\frac{1}{(\alpha t)^{1/2}}$$

$$\frac{\partial^2 T^*}{\partial x^2} = f''\frac{1}{\alpha t}$$

$$\frac{\partial T^*}{\partial t} = f'\frac{\partial \eta}{\partial t} = f'\left(-\frac{1}{2}\frac{\eta}{t}\right)$$

$$\therefore\ -\tfrac{1}{2}\eta f' = f'',\quad f(0)=1,\quad f(\infty)=0$$

Integrating once:

$$f' = C_1 e^{-\eta^2/4}$$

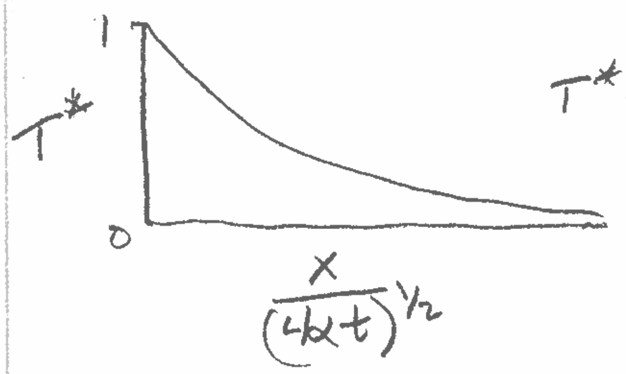
$$f = -\int_\eta^\infty C_1 e^{-\eta^2/4}\,d\eta \qquad \text{(satisfies B.C. at } \infty)$$

$$1 = -C_1\int_0^\infty e^{-\eta^2/4}\,d\eta$$

$$\therefore\ f = \frac{\int_\eta^\infty e^{-\eta^2/4}\,d\eta}{\int_0^\infty e^{-\eta^2/4}\,d\eta} = 1 - \frac{\int_0^\eta e^{-\eta^2/4}\,d\eta}{\int_0^\infty e^{-\eta^2/4}\,d\eta}$$

$$= 1 - \mathrm{erf}\left(\frac{\eta}{2}\right) \quad \text{or} \quad \mathrm{erfc}\left(\frac{\eta}{2}\right)$$

T profile looks like:



$T^* = .01$ at $\frac{x}{(4\alpha t)^{1/2}} \approx 2$

$\therefore$ thermal penetration length $\delta_T \approx 4(\alpha t)^{1/2}$

What is energy flux?

$$q = -K \frac{\partial T}{\partial x}\bigg|_{x=0} = \frac{K}{\sqrt{\pi \alpha t}}(T_1 - T_0)$$

which diverges for small t, as would be expected.

What happens if we have a finite slab?
(BS&L 11.1)

still have $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$, but now have B.C.'s

$T = T_0$ at $t = 0$, $T = T_1$ at $x = \pm b$

have length scale b, time scale b^2/α

$$\therefore T^* = \frac{T_1 - T}{T_1 - T_0} = f^n\left(\frac{x}{b}, \frac{t\alpha}{b^2}\right)$$

thus $\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^{*2}}$ w/ I.C. $T^* = 1$ and $T^* = 0$ at $x^* = \pm 1$

Stretching: B.C. at ± 1 wipes out sim. sol'n

Use sep. of variables:

$T^* = f(x^*)g(t^*) \Rightarrow$ drop $*$'s :

$$\frac{g'}{g} = \frac{f''}{f} = -c^2 \leftarrow \text{require decaying sol}^{\text{n}}$$

$$g = e^{-c^2 t}, \quad f = B \sin cx + D \cos cx$$

f must be symmetric from B.C.'s $\therefore B = 0$

cond. at $x = \pm 1$ gives eigenvalues:

$$c_n = (n + \tfrac{1}{2})\pi$$

$$\therefore T^* = \sum_{n=0}^{\infty} D_n e^{-(n+\frac{1}{2})^2 \pi^2 t^*} \cos((n+\tfrac{1}{2})\pi x^*)$$

where $D_n = \dfrac{\int_{-1}^{1} \overset{T^*|_{t=0}=1}{\cos}((n+\frac{1}{2})\pi x)\, dx}{\int_{-1}^{1} \cos^2((n+\frac{1}{2})\pi x)\, dx}$

or $D_n = \dfrac{2(-1)^n}{(n+\frac{1}{2})\pi}$

and $T^* = \dfrac{T_1 - T}{T_1 - T_0} = 2 \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\frac{1}{2})\pi} e^{-(n+\frac{1}{2})^2 \pi^2 \frac{\alpha t}{b^2}} \cos\left[(n+\tfrac{1}{2})\frac{\pi x}{b}\right]$

this series converges very slowly for small $\frac{\alpha t}{b^2}$:

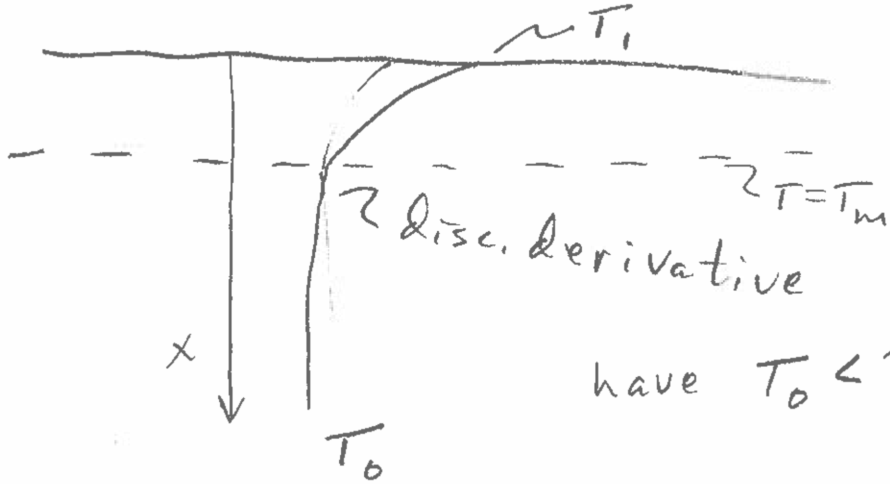
End Lec. 11

in this region more appropriate to examine semi-inf slab.

One-dim. E cond. still area of active res: in Camb. studying melting in lava chamber.

Question: does lava melt surrounding rock or does lava solidify? ⇒ complex, f$^{\underline{n}}$. turbulent transp, mixing, etc.

Look at simpler problem:



have $T_0 < T_m < T_1$

Assume cst α, K (both liquid & solid)

h t. of fusion L (per unit volume)

wish to determine $x_m(t)$ ⇒ advance of melt zone.

Prob. which arises in zone refining of Si, etc.

Indep. Parameters:

$$T_1 - T_0,\ T_M - T_0,\ L,\ \alpha,\ K$$

indep. var: x, t

dep. var: $T - T_0$, X_m

Look at units:

$$[T_1 - T_0], [T_M - T_0] = {}^0K,\ [L] = E/L^3,\ [K] = \frac{E}{LT\,{}^0K},\ [\alpha] = \frac{L^2}{T}$$

indep. param.

Form dimensional matrix:

$$\begin{array}{c|ccccc} & \Delta T & L & K & \alpha & \Delta T_m \\ \hline E & 0 & 1 & 1 & 0 & 0 \\ L & 0 & -3 & -1 & 2 & 0 \\ T & 0 & 0 & -1 & -1 & 0 \\ {}^\circ K & 1 & 0 & -1 & 0 & 1 \end{array}$$

sub. add sub.

not zero for matrix inc. x_m, t

$\therefore x_m, t, \Delta T, \Delta T_m, L, K, \alpha$

$7-4=3$

$$\text{Now det}\left|\quad\right| = \det\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 0$$

Cannot form ref. length scale or time scale from indep. param! $7-4=3$ π groups

$$\therefore x_M = (\alpha t)^{1/2} f^n\left(\frac{T_m - T_0}{T_1 - T_0}, \frac{L\alpha}{K(T_1 - T_0)}\right)$$

and we have $x_M \sim t^{1/2}$ from dim. anal.

and expect sim. sol'n:

$$\frac{T - T_0}{T_1 - T_0} = f^n\left(\eta, \frac{T_m - T_0}{T_1 - T_0}, \frac{L\alpha}{k(T_1 - T_0)}\right)$$

where $\eta = \frac{x}{(\alpha t)^{1/2}}$ melt given by $\eta = \eta_m = cst$

Problem is governed by P.D.E.:

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}, \quad x \neq x_m \quad \text{w/ I.C. } T = T_0 \text{ at } t = 0,$$

$$\text{B.C.'s} \quad T\Big|_{x=0} = T_1, \quad T\Big|_{x=\infty} = T_0$$

and conditions at melt line:

T ≡ continuous at $x = x_m$,
jump cond. at melting pt.

$$\left[-k\frac{\partial T}{\partial x}\right]\bigg|_{x=x_m-\delta} - \left[-k\frac{\partial T}{\partial x}\right]\bigg|_{x=x_m+\delta} = L\frac{dx_m}{dt}$$

Let $T^* = \dfrac{T-T_0}{T_1-T_0}$, $\zeta = \dfrac{x}{(\alpha t)^{1/2}}$, $M = \dfrac{k(T_1-T_0)}{L\alpha}$

for $\zeta \neq \zeta_m$, $-\frac{1}{2}\zeta\dfrac{dT^*}{d\zeta} = \dfrac{d^2T^*}{d\zeta^2}$

at $\zeta = \zeta_m$, $x_m = \zeta_m(\alpha t)^{1/2}$

$\therefore \dfrac{dx_m}{dt} = \frac{1}{2}\zeta_m\left(\dfrac{\alpha}{t}\right)^{1/2}$

and jump condition at ζ_m becomes:

$$\zeta_m = 2M\left(\frac{dT^*}{d\zeta}\bigg|_{\zeta_m+\delta} - \frac{dT^*}{d\zeta}\bigg|_{\zeta_m-\delta}\right)$$

w/ B.C.'s $T^* = 1$ at $\zeta = 0$ and $T^* = 0$ at $\zeta = \infty$
To obtain sol'n, solve in region $0 < \zeta < \zeta_m$, $\zeta_m < \zeta < \infty$ & det. ζ_m by matching condition.

for $\zeta < \zeta_m$:

$$\frac{dT^{I}}{d\zeta} = c_1 e^{-\zeta^2/4}, \quad T^{I}(0) = 1$$

$$\therefore T^{I} = 1 + \int_0^{\zeta} c_1 e^{-y^2/4} dy$$

$$= 1 + c_1\sqrt{\pi}\, \mathrm{erf}(\zeta/2)$$

for $\zeta > \zeta_m$:

$$\frac{dT^{II}}{d\zeta} = c_2 e^{-\zeta^2/4}, \quad T^{II}(\infty) = 0$$

$$\therefore T^{II} = -\int_{\zeta}^{\infty} c_2 e^{-y^2/4} dy = -c_2\sqrt{\pi}\left(1 - \mathrm{erf}(\zeta/2)\right)$$

at $\zeta = \zeta_m$, $T^{I} = T^{II} = T_m^*$

$$\therefore c_1 = -\frac{1 - T_m^*}{\sqrt{\pi}\, \mathrm{erf}(\zeta_m/2)}$$

$$c_2 = -\frac{1}{\sqrt{\pi}} \frac{T_m^*}{1 - \mathrm{erf}(\zeta_m/2)}$$

and the jump condition:

$$\zeta_m = 2M\left(c_2 e^{-\zeta_m^2/4} - c_1 e^{-\zeta_m^2/4}\right)$$

$$\zeta_m = M \frac{2}{\sqrt{\pi}} e^{-\zeta_m^2/4} \left(\frac{1 - T_m^*}{\operatorname{erf}(\zeta_m/2)} - \frac{T_m^*}{1 - \operatorname{erf}(\zeta_m/2)} \right)$$

which is an implicit equation for $\zeta_m(M, T_m^*)$.
We may evaluate two limits: $M \to 0$ (lge L) and $M \to \infty$ (small L)

For large M, jump cond. gives:

$$C_2 - C_1 = \frac{1}{M} \zeta_m e^{\zeta_m^2/4} \approx 0$$

provided $\zeta_m = O(1)$; i.e. $T_m^* \sim \frac{1}{2}$

thus if $C_2 = C_1$, have sol$^{\text{n}}$ w/out melting:

$$\underline{T_m^* = 1 - \operatorname{erf}(\zeta_m/2)}$$

More interesting for $M \to 0$.

In this case, ζ_m will be small, thus

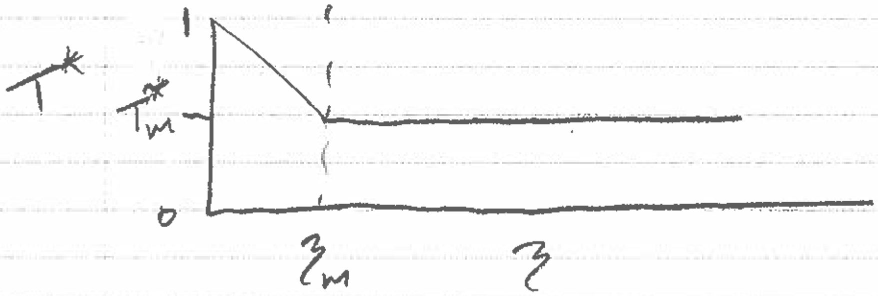
$$\operatorname{erf}(\zeta_m/2) \approx \frac{1}{\sqrt{\pi}} \zeta_m + O(\zeta_m^2)$$

and
$$\zeta_m \approx M \frac{2}{\sqrt{\pi}} \left(\frac{1 - T_m^*}{\frac{1}{\sqrt{\pi}} \zeta_m} - \cancel{\frac{T_m^*}{1}} \right)$$
small

$$\therefore \zeta_m \approx \left(2M(1 - T_m^*)\right)^{1/2}$$

or $x_m = \left(2 \frac{kt}{L}(T_1 - T_m)\right)^{1/2}$

which corresponds to the temp. profile:



can recover latter result physically:

$$q\Big|_{x=x_m} = -k\frac{\partial T}{\partial x}\Big|_{x=x_m} \approx -k\frac{T_m - T_1}{x_m} = L\frac{dx_m}{dt}$$

$$\therefore\ x_m = \left(2\frac{kt}{L}(T_1 - T_m)\right)^{1/2}$$

Analogous problem for cst. q does not admit sim. sol'n;

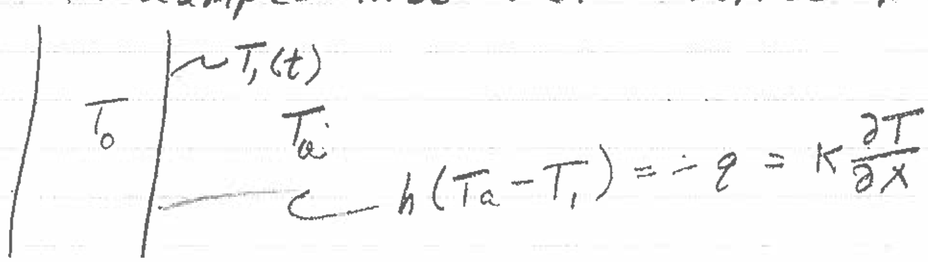
(Talk about asymp method ⇒ skip 2 pgs)

~~for short time~~,

Return to finite slab: most realistic B.C. for quenching not $T_1 = cst$, but rather

$$h(T_a - T_1) = -k\frac{\partial T}{\partial x}\Big|_{x=b} \Rightarrow \text{ht. transf. coeff.}$$

↳ B.C. if dumped into well-stirred bath at T_a



let $T^* = \frac{T_a - T}{T_a - T_o}$, $x^* = \frac{x}{a}$, $t^* = \frac{\alpha t}{b^2}$

have P.D.E. $\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial y^{*2}}$

w/ I.C. $T^* = 1$ at $t^* = 0$

& B.C. $\frac{\partial T^*}{\partial y^*}\left(\frac{\kappa}{b}\right) = h T^*$ at $y^* = 1$

or $\frac{\partial T^*}{\partial y^*} = \underbrace{\left(\frac{bh}{k}\right)}_{\text{Biot \#} \equiv \frac{\text{internal resis}}{\text{surface resis.}}} T^*$ at $y^* = 1$

If $Bi >> 1$, B.C. reduces to $T^* = 0$ at $y^* = 1$ = same problem already solved!

If $Bi = 0$, obtain $\left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=1} = 0$ and $T^* = 1$ everywher just an insulated slab!

For small Bi, obtain uniform temp in slab, T^* decreases by ht. transf thru film:

$$b \rho \hat{C}_p \frac{dT}{dt} = -h(T_a - T)$$

End lec. 12

$$\therefore \frac{dT^*}{dt^*} = -Bi\,T^*, \quad T^* = e^{-Bi\,t^*} = e^{-\frac{ht}{\rho \hat{C}_p b}}$$

To solve for all Bi, use sep. of variables:

$T^* = f(x^*)\, g(t^*)$ (dropping *'s):

$$\frac{f''}{f} = \frac{g'}{g} = -c^2, \quad g = e^{-c^2 t}$$

$$f = A \cos cx + \cancel{B} \sin cx$$

$\qquad\qquad 0$ as require $\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$

$$\therefore\ T = A e^{-c^2 t} \cos cx, \quad \frac{\partial T}{\partial x} = -cAe^{-c^2 t} \sin cx$$

at $x = 1$:

$$\frac{\partial T}{\partial x} = -B_i T \quad \therefore\ \cancel{\left.(\sin cx)\right|_{x=1} = c\, B_i \cos cx|_{x=1}}$$

or $\cancel{c_n \text{ s.t. } \frac{\tan c_n}{c_n} = B_i}$ $\qquad$ c_n s.t. $c_n \tan c_n = B_i$

$\left. c_0 \right|_{B_i \to 0} = B_i^{1/2}$

$\left. c_0 \right|_{B_i \to \infty} = \frac{\pi}{2}$

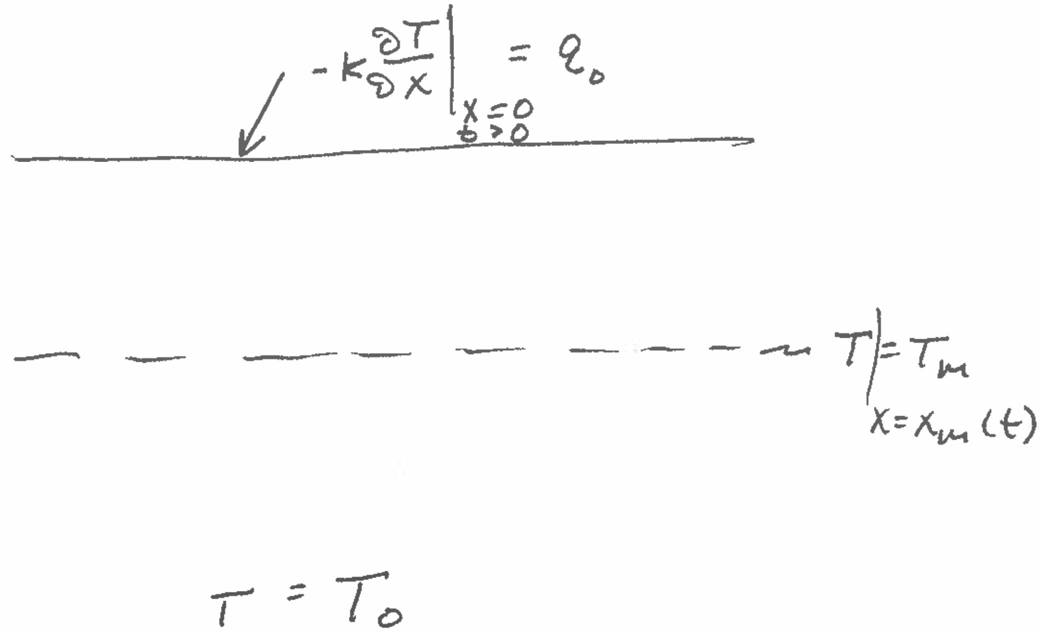
thus

$$T^* = \sum_{n=1}^{\infty} A_n e^{-c_n^2 t^*} \cos c_n x^*$$

where $A_n = \dfrac{2 \sin c_n}{c_n + \sin c_n \cos c_n}$

↳ obtain from orthogonality & I.C.

Heat conduction w/ melting : constant heat flux



$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad -k\frac{\partial T}{\partial x}\Big|_{x=0} = q_0$$

$$T\Big|_{x=\infty} = T_0$$

$$T\Big|_{x=x_m(t)} = T_M$$

$$-k\frac{\partial T}{\partial x}\Big|_{x=x_m-\delta} + k\frac{\partial T}{\partial x}\Big|_{x=x_m+\delta} = L\frac{dx_m}{dt}$$

Let $T^* = \dfrac{T - T_0}{T_M - T_0}$

$$x^* = \frac{x}{\left(\frac{k(T_M - T_0)}{q_0}\right)} \qquad t^* = \frac{\alpha t}{\left(\frac{k(T_M - T_0)}{q_0}\right)^2}$$

$$\therefore \frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^{*2}}$$

$$T^*\Big|_{t^*=0} = 0 \qquad \frac{\partial T^*}{\partial x^*}\Big|_{\substack{x^*=0 \\ t^*>0}} = -1$$

$$T^*\Big|_{x^* = X_M^*(t^*)} = 1 \quad ;$$

$$-\frac{\partial T^*}{\partial x^*}\Big|_{x^* = X_M^* - \delta} + \frac{\partial T^*}{\partial x^*}\Big|_{x^* = X_M^* + \delta} = \left[\frac{L\alpha}{k(T_M - T_0)}\right]\frac{dX_M^*}{dt^*}$$

Thus the dimensionless problem is a function of a <u>single</u> dimensionless parameter. <u>But</u> it will no longer admit a similarity solution!

in the limit $\frac{L\alpha}{k(T_m - T_0)} = \infty$ we have the simplified problem:

$$L \frac{dx_m}{dt} = q_0$$

$$\therefore\ x_m = \frac{q_0}{L} t$$

or: $$x_m^* = \frac{k(T_m - T_0)}{L\alpha} t^*$$

In the limit that $\frac{L\alpha}{k(T_m - T_0)} = 0$ we have the problem without melting:

$$\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^{*2}} \ ; \quad T^*\Big|_{t^*=0} = 0 \ , \quad \frac{\partial T^*}{\partial x^*}\Big|_{\substack{x^*=0 \\ t^*>0}} = -1$$

Thus we have the solution:

$$T^* = t^{*\,1/2}\, 2\, \mathrm{ierfc}\left(\frac{x^*}{(4t^*)^{1/2}}\right)$$

where x_m^* is determined from the implicit relationship:

$$1 = t^{*1/2}\, 2\, \mathrm{ierfc}\left(\frac{x_m^*}{(4t^*)^{1/2}}\right)$$

Note that this will have no solution for short times $t^* < t_c^*$. This is because $T^* < T_m^*$ even for $x_m^* = 0$

This critical time is given by:

$$1 = t_c^{*1/2}\, 2\, \mathrm{ierfc}(0) = t_c^{*1/2}\, \frac{2}{\sqrt{\pi}}$$

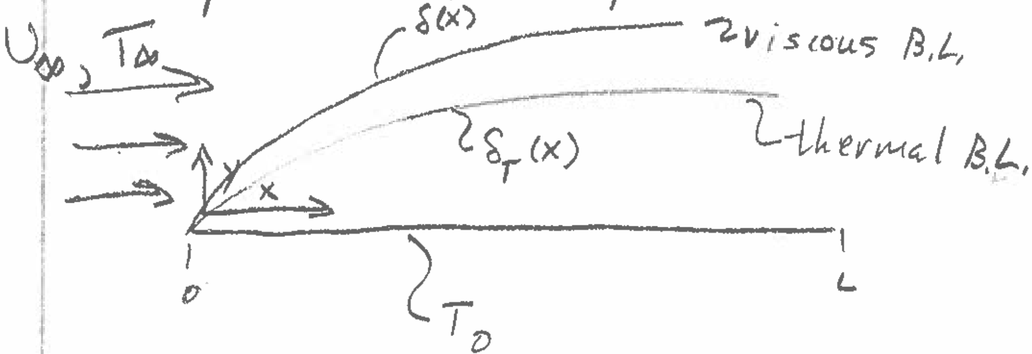
$$\therefore\ t_c^{*1/2} = \frac{\sqrt{\pi}}{2}$$

which is the delay before the surface begins to melt.

Steady-state B.L. theory

(B.S.&L 11.4, Schlichting XII g.)

Flow past a flat plate at zero incidence:



Assume cst properties & no viscous dissip.

(latter assumption may not be good in viscous B.L. due to high shear)

CE $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \quad$ (C.E.)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{1}{\rho}\frac{\partial P}{\partial x}$$

MOM

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{1}{\rho}\frac{\partial P}{\partial y} - g$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

If L is char. Length, U_∞ char. velocity, obtain dimensionless eq'ns $\left(T^* = \dfrac{T - T_0}{T_\infty - T_0}\right)$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{Re}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) - \frac{\partial p^*}{\partial x^*}$$

where $Re = \frac{U_\infty L}{\nu}$, $p^* = \frac{p}{\rho U_\infty^2}$

and

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = \frac{1}{Re}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right) - \frac{\partial p^*}{\partial y^*} - \frac{1}{Fr}$$

where $Fr = \frac{U_\infty^2}{gL}$ => can elim. by redefining p.

$$P = p + \rho g y$$

and

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{Re Pr}\left[\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}}\right]$$

Non-linear P.D.E., but <u>not</u> coupled : may solve for u^*, v^* & then obtain T^*

If $Re \gg 1$, tempted to throw out terms $O(\frac{1}{Re})$ & $O(\frac{1}{Re Pr})$

If we do, get Euler equations (potential flow)

$\rho (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}) \underset{\sim}{u} = -\underset{\sim}{\nabla} P$ => first order, ∴ <u>cannot</u> satisfy no-slip B.C. at plate. Sol'n is just $u^* = 1$ everywhere (applying cond. at $y^* = \infty$). For small y^*, large change in u^*, thus <u>must</u> be some B.L. where viscous forces are important.

Re-scale y:

let $y^* = y/\delta$, $v^* = v/V_0 \leftarrow$ char. velocity in y dir.

from C.E:

$$\frac{\partial u^*}{\partial x^*} + \left(\frac{L}{\delta}\frac{V_0}{U_\infty}\right)\frac{\partial v^*}{\partial y^*} = 0$$

terms <u>must</u> be of same order, ∴ let:

$$\frac{L}{\delta}\frac{V_0}{U_\infty} = 1, \text{ or } V_0 = U_\infty \frac{\delta}{L}$$

where we anticipate $\delta/L \ll 1$

Putting this scaling into x-mom. eq'n:

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{1}{Re}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \left(\frac{L}{\delta}\right)^2\frac{\partial^2 u^*}{\partial y^{*2}}\right) - \frac{\partial p^*}{\partial x^*}$$

For large Re, anticipate $L/\delta \gg 1$ ∴ y diff. >> x-diff.

We wish to balance convection w/ dif$^{\underline{n}}$ ∴ let

$$\left(\frac{L}{\delta}\right)^2\frac{1}{Re} = 1 \quad \therefore \quad \frac{\delta}{L} = \underbrace{Re}_{\text{based on } L}^{-1/2}$$

Now for y-mom. eq'n:

$$\frac{\delta}{L}\left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) = \frac{1}{Re}\frac{\delta}{L}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \left(\frac{L}{\delta}\right)^2\frac{\partial^2 v^*}{\partial y^{*2}}\right) - \frac{L}{\delta}\frac{\partial p^*}{\partial y^*}$$

Or $\frac{\partial p^*}{\partial y^*} = O\left(\frac{1}{Re}\right) \cong 0$

or pressure outside B.L. is <u>impressed</u> on B.L.

Now for energy eq'n:

Unless $Pr = 1$, thermal B.L. will have different thickness than mom. B.L.; i.e. will be $f^{\underline{n}}(Pr)$

let $y_T^* = y/\delta_T$, leave velocity scaling unchanged:

$$u^* \frac{\partial T^*}{\partial x^*} + \left(\frac{\delta}{\delta_T}\right) v^* \frac{\partial T^*}{\partial y_T^*} = \frac{1}{Re\,Pr}\left(\frac{\partial^2 T^*}{\partial x^{*2}} + \underbrace{\frac{L^2}{\delta_T^2}\frac{\partial^2 T^*}{\partial y^{*2}}}_{\text{dominates conduction}}\right)$$

Thus, $\frac{1}{Re\,Pr}\frac{L^2}{\delta_T^2} = 1$

or $\delta_T/L = \left(\frac{1}{Re\,Pr}\right)^{1/2}$ and $\delta/\delta_T = Pr^{1/2}$

w/ $u^* \frac{\partial T^*}{\partial x^*} + Pr^{1/2} v^* \frac{\partial T^*}{\partial y_T^*} = \frac{\partial^2 T^*}{\partial y_T^{*2}}$

still contains Pr $\therefore$ sol'n <u>will</u> be $f^{\underline{n}}(Pr$

If $Pr = 1$, $\delta \approx \delta_T$ (gases)

if $Pr \approx 10$ (H_2O $Pr = 7$), $\delta_T \approx \frac{1}{3}\delta$ $\Rightarrow$ mom. diffuses faster into fluid

for liquid metals, $Pr \ll 1$ and ht. diffuses faster — this is problem solved for HW (uniform flow past a hot plate)

<u>End lec. 13</u>

We have the scaled eq'ns:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

note similarity

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$ (assume $\frac{\partial P}{\partial x}=0$ — flat plat

$$u\frac{\partial T^*}{\partial x}+v\frac{\partial T^*}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^*}$$

Assoc. B.C.'s: $y=0$: $u=v=T^*=0$, $T^*=\frac{T-T_0}{T_\infty-T_0}$

$y=\infty$: $u=u_\infty$, $T^*=1$

Now look at case of semi-infinite flat plate (No length scale L)

Can construct ref. length from remaining param (i.e., ν/u_∞), but expect dif. length for x, y dir.

$\therefore$ expect similarity transform.

Introduce stream f^n ψ:

$$u=\frac{\partial \psi}{\partial y},\quad v=-\frac{\partial \psi}{\partial x}$$

$\therefore$ C.E. satisfied,

x-mom: $\psi_y\psi_{xy}-\psi_x\psi_{yy}=\nu\psi_{yyy}$

at $y=0$, $\psi_x=\psi_y=0$; $y=\infty$ $\psi_y=u_\infty$

To obtain sim. transf., try stretching:

Let: $\psi = A\bar{\psi}$, $x = B\bar{x}$, $y = C\bar{y}$

$$\therefore \frac{A^2}{BC^2}\left(\overline{\psi_y \psi_{xy} - \psi_x \psi_{yy}}\right) = \nu \frac{A}{C^3}\overline{\psi_{yyy}}$$

or $\frac{AC}{B} = 1$

w/ cond. at $\bar{y} = \infty$, $\frac{A}{C}\bar{\psi}_y = u_\infty$

$\therefore \frac{C}{B^{1/2}} = 1$, $\frac{A}{B^{1/2}} = 1$

Sim. Rule: $\psi = (u_\infty \nu)^{1/2} \underline{x^{1/2}} f(\eta)$

$\eta = \sqrt{\frac{u_\infty}{\nu}} \frac{y}{x^{1/2}}$ where csts chosen to leave η, $f(\eta)$ dimensionless

Plugging in, get Blasius eq'n:

$$2f''' + ff'' = 0 \quad \text{w/ B.C.'s } f(0) = f'(0) = 0 \text{ and } f'(\infty) = 1$$

Energy eq'n also admits sim. sol'n:
$T^* = f^n(\eta)$

$$\therefore T^{*''} + \frac{Pr}{2} f T^{*'} = 0 \quad \text{w/ } T^*(0) = 0, T^*(\infty) =$$

Non-linear O.D.E. has no known closed form sol'n, so must be solved numerically.

Series sol'n:

$$f = \frac{a\eta^2}{2!} - \frac{1}{2}\frac{a^2\eta^5}{5!} + \frac{11}{4}\frac{a^3\eta^8}{8!} - \frac{375}{8}\frac{a^4\eta^{11}}{11!} + \ldots$$

where $a \approx 0.332$ from numerical result

Note that $v \neq 0$ as $y \to \infty$: due to displacement of streamlines caused by B.L.

Also, $u/u_\infty \cong 0.99$ at $\eta = 5.0$

$$\therefore \frac{\delta}{x} \cong \frac{5.0}{\sqrt{\frac{u_\infty x}{\nu}}} = \frac{5.0}{Re_x^{1/2}} \Big\} \text{thickness of B.L.}$$

Given f, may determine T^*:

$$T^{*''} = -\frac{Pr}{2} f(\eta)\, T^{*'}$$

$$\therefore T^{*'} = c_1 e^{-\int_0^\eta \frac{Pr}{2} f(s)\,ds}$$

where $c_1 = T^{*'}(0)$

and $$T^* = \frac{\int_0^\eta \exp\left[-\int_0^r \frac{Pr}{2} f(s)\,ds\right] dr}{\int_0^\infty \exp\left[-\int_0^r \frac{Pr}{2} f(s)\,ds\right] dr} \Bigg\} \text{applied B.C.'s at } \eta = 0, \infty$$

where $$T^{*'}(0) = c_1 = \left[\int_0^\infty \exp\left[-\int_0^r \frac{Pr}{2} f(s)\,ds\right]\right]^{-1}$$

must be integrated numerically:

$$\left.\frac{\partial T^*}{\partial \eta}\right|_{\eta=0} \approx 0.332\, Pr^{0.343} \qquad \text{for } Pr < 0.5$$

can use result to obtain Nu_x:

↳ local Nu

$$Nu_x = h_x \cdot \frac{x}{k}$$

↳ defined by custom: physically should be δ, but that is unknown.

$$h_x = \frac{-k \left.\frac{\partial T}{\partial y}\right|_{y=0}}{T_0 - T_\infty} = k\sqrt{\frac{u_\infty}{\nu x}}\left.\frac{\partial T^*}{\partial \eta}\right|_{\eta=0}$$

→ actually ~ .343 (complex fn)

$$\therefore Nu_x \approx 0.332\, Pr^{1/3}\, Re_x^{1/2}$$

exact

or ave. value:

$$Nu_L \approx 0.664\, Pr^{1/3}\, Re_L^{1/2} = \frac{1}{L}\int_0^L Nu_x\, dx$$

For very low Pr:

$$Nu_x \approx \frac{0.564\sqrt{Re_x Pr}}{1 + 0.90\sqrt{Pr}} \qquad 0.006 < Pr < 0.03$$

Result from Sparrow & Greg

BS&L uses semi-integral approach to obtain approx. sol'n (11.4-1) ⇒ review & contrast w/ approach used here.

Rescaling of Thermal B.L. eq'ns

We have the equations:

$$2f''' + ff'' = 0 \qquad f(0) = f'(0) = 0$$
$$f'(\infty) = 1$$

where $\psi = (U_\infty \nu)^{1/2} x^{1/2} f(\eta)$

$$\eta = \left(\frac{U_\infty}{\nu}\right)^{1/2} \frac{y}{x^{1/2}}$$

and:

$$u = \frac{\partial \psi}{\partial y} = U_\infty f'(\eta)$$

This will have a solution where:

$$f\Big|_{\eta \to \infty} = \eta + O(1)$$

and

$$f\Big|_{\eta \to 0} = \frac{0.332}{2} \eta^2 + O(\eta^5)$$

from a series solution.

The Energy equation is:

$$\frac{d^2 T^*}{d\eta^2} + \frac{Pr}{2} f \frac{dT^*}{d\eta} = 0$$

with B.C.'s $T^*\big|_{\zeta=0} = 0 \quad T^*\big|_{\zeta=\infty} = 1$

we require the two terms to be in balance for all Pr. Thus it is reasonable to rescale ζ for the thermal B.L.

We first look at the limit $Pr \to 0$. Here we expect the thermal B.L. to be much thicker than the momentum B.L.

Thus, we let:

$$\zeta_T = \zeta \, Pr^{n}$$

where n is to be determined. The new variable ζ_T is $O(1)$ in the thermal B.L.

Substituting:

$$\frac{d^2T^*}{d\zeta_T^2} + \frac{Pr^{1-n}}{2} f\left(\frac{\zeta_T}{Pr^{n}}\right)\frac{dT^*}{d\zeta_T} = 0$$

Now since $\zeta_T/Pr^{n} \gg 1$ we substitute in the asymptotic form for $f(\zeta)$:

$$f\left(\zeta_T/Pr^{n}\right) \cong \frac{\zeta_T}{Pr^{n}} + O(1)$$

Thus:

$$\frac{d^2 T^*}{d \zeta_T^2} + \frac{P_r^{1-2n}}{2} \zeta_T \frac{dT^*}{d\zeta_T} \simeq 0$$

The two terms are of the same order when $n = \frac{1}{2}$!

Thus:

$$\frac{d^2 T^*}{d \zeta_T^2} + \frac{1}{2} \zeta_T \frac{dT^*}{d\zeta_T} \simeq O\left(P_r^{-1/2}\right)$$

which is the same as unsteady conduction into a half-space:

$$T^* = \text{erf}\left(\zeta/2\right)$$

Now for the limit $P_r \to \infty$

Again we rescale ζ, only now the thermal B.L. is very <u>thin</u>. Thus:

$$\zeta_T = \zeta P_r^{\,n}$$

$$\frac{d^2T^*}{d\zeta_T^2} + \frac{P_r^{1-n}}{2} f\left(\frac{\zeta_T}{P_r^{n}}\right)\frac{dT^*}{d\zeta_T} = 0$$

In this case we seek the behavior of f as $\zeta \to 0$:

$$f = \frac{0.332}{2}\zeta^2 + O(\zeta^5)$$

$$= \frac{0.332}{2} P_r^{-2n} \zeta_T^2 + O\left(P_r^{-5n}\right)$$

$$\therefore \frac{d^2T^*}{d\zeta_T^2} + P_r^{1-3n}\left(\frac{0.332}{4}\right)\zeta_T^2 \frac{d^2T^*}{d\zeta_T} = O\left(P_r^{-5n}\right)$$

To balance, we require $n = \frac{1}{3}$

Thus:

$$\frac{d^2T^*}{d\zeta_T^2} + \left(\frac{0.332}{4}\right)\zeta_T^2 \frac{d^2T^*}{d\zeta_T} = O\left(P_r^{-5/3}\right)$$

This will have as its solution the incomplete Γ function.

Domain of validity for B.L. equations

We had the equations:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Let $x^* = x/L$, $y^* = y/\delta$, $u^* = u/U$, $v^* = \frac{v}{\frac{\delta}{L}U}$

$$\therefore u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{L^2}{\delta^2}\frac{1}{Re}\left(\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\delta^2}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}}\right)$$

Choose $\delta = L\frac{1}{Re^{1/2}}$

$\therefore$ Ignore $\frac{\partial^2 u^*}{\partial x^{*2}}$ if $Re >> 1$

$$\frac{UL}{\nu} >> 1$$

Now we require x^* to be $O(1)$ in region of interest. Thus the B.L. equations will apply provided:

$$\frac{Ux}{\nu} >> 1 \quad \text{or} \quad x >> \frac{\nu}{U}$$

What about the thermal B.L. eq'ns?

In this case we have:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

For $Pr \gg 1$ we <u>still</u> have the same domain of validity, because the u, v sol'ns are only good for

$$\frac{Ux}{\nu} \gg 1$$

even though we can neglect $\frac{\partial^2 T}{\partial x^2}$ at smaller x! For $Pr \ll 1$ we have:

$$\frac{Ux}{\alpha} \gg 1$$

for validity. Thus we require:

$$\frac{Ux}{\nu} \gg \max\left(1, \frac{1}{Pr}\right)$$

for our similarity solution to hold!