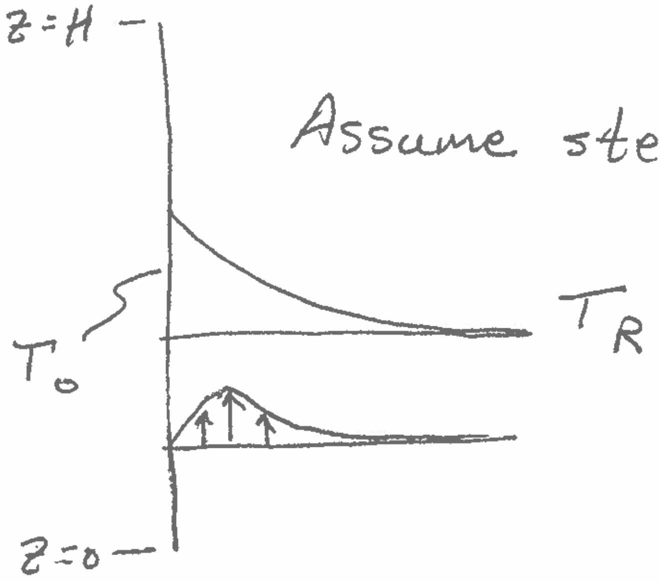
Free Convection Past a Flat Plate: Scaling of Equations

Assume steady state



C.E. : $\underset{\sim}{\nabla}\cdot(\rho\underset{\sim}{u})=0$

$$\rho\,\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\underset{\sim}{u}=-\underset{\sim}{\nabla}p+\rho\underset{\sim}{g}+\underset{\sim}{\nabla}\cdot\underset{\approx}{\tau}$$

$$\underset{\approx}{\tau}=\mu\left(\underset{\sim}{\nabla}\underset{\sim}{u}+(\underset{\sim}{\nabla}\underset{\sim}{u})^T-\frac{2}{3}\underset{\approx}{\delta}\,\underset{\sim}{\nabla}\cdot\underset{\sim}{u}\right)$$

$$\underset{\sim}{u}\cdot\underset{\sim}{\nabla}T=\frac{1}{\rho\hat{C}_p}\underset{\sim}{\nabla}\cdot\left(k\underset{\sim}{\nabla}T\right)$$

We seek a solution for small $\Delta T\equiv T_0-T_R$

Since $\underset{\sim}{u}=0$ if $\Delta T=0$ we expect that $\underset{\sim}{u}$ will vary as ΔT^n where n is some positive number. We may thus

Do a perturbation expansion in small ΔT for the quantities ρ, k, and μ and keep only the leading terms. We also define the augmented pressure:

$$\mathcal{P} = p - \rho_R \, \underset{\sim}{g} \cdot \underset{\sim}{x}$$

where $\mathcal{P}$ will also be zero if $\Delta T = 0$

We let $\rho = \rho_R + \left.\frac{\partial \rho}{\partial T}\right|_{T_R} T^* \Delta T + O(\Delta T^2)$

$$\mu = \mu_R + \left.\frac{\partial \mu}{\partial T}\right|_{T_R} T^* \Delta T + O(\Delta T^2)$$

$$k = k_R + \left.\frac{\partial k}{\partial T}\right|_{T_R} T^* \Delta T + O(\Delta T^2)$$

where $T^* = \frac{T - T_R}{T_0 - T_R}$

Thus:

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} + O\big(\underbrace{u \Delta T}_{\text{small}}\big) = 0$$

so $\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0$ to leading order

Similarly,

$$\underset{\approx}{\tau} = \mu_R \left(\underset{\sim}{\nabla} \underset{\sim}{u} + (\underset{\sim}{\nabla} \underset{\sim}{u})^T - \frac{2}{3} \underset{\approx}{\delta} \underbrace{\cancel{\underset{\sim}{\nabla} \underset{\sim}{u}}}_{0 \text{ from C.E.}} \right)$$

$$\rho \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \underset{\sim}{u} \cong \rho_R \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \underset{\sim}{u}$$

$$-\underset{\sim}{\nabla} p + \rho \underset{\sim}{g} = -\underset{\sim}{\nabla} \mathcal{P} + (\rho - \rho_R) \underset{\sim}{g}$$

$$= -\underset{\sim}{\nabla} \mathcal{P} + \underset{\sim}{g} \frac{\partial \rho}{\partial T} \Big|_{T_R} \Delta T \, T^* + O(\Delta T^2)$$

Let $-\frac{\partial \rho}{\partial T} \Big|_{T_R} \frac{1}{\rho_R} \equiv \beta$

$$\therefore \; -\underset{\sim}{\nabla} p + \rho \underset{\sim}{g} \cong -\underset{\sim}{\nabla} \mathcal{P} - \beta \underset{\sim}{g} T^* \Delta T$$

To leading order in ΔT we have the equations:

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = O\left(\beta \Delta T \frac{U}{L} \right)$$

$$\underset{\sim}{u} \cdot \underset{\sim}{\nabla} \underset{\sim}{u} = -\frac{1}{\rho_R} \underset{\sim}{\nabla} \mathcal{P} - \beta \underset{\sim}{g} T^* \Delta T + \frac{\mu_R}{\rho_R} \nabla^2 \underset{\sim}{u}$$

$$+ O\left(\Delta T \left(\frac{\mu'}{\mu_R} \frac{U}{L^2}, \beta \frac{U^2}{L} \right) \right)$$

Finally, we have the energy equation:

$$\underset{\sim}{u} \cdot \underset{\sim}{\nabla} T^* = \alpha_R \nabla^2 T^* + O\left(\alpha_R \frac{\Delta T}{L^2} (\beta, \frac{K'}{K_R})\right)$$

where L is a characteristic length scale.

Subject to a posteriori justification we will ignore all of these higher order terms.

OK, now we write the problem out in 2-D form:

$$\underset{\sim}{u} \equiv u \, \underset{\sim}{\hat{e}}_x + v \, \underset{\sim}{\hat{e}}_y + w \, \underset{\sim}{\hat{e}}_z$$

$$\underset{\sim}{g} = -g \, \underset{\sim}{\hat{e}}_z$$

$$\therefore \quad \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho_R} \frac{\partial P}{\partial z} + \beta g T^* \Delta T + \frac{\mu_R}{\rho_R} \left(\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial y^2} \right)$$

$$v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho_R} \frac{\partial P}{\partial y} + \frac{\mu_R}{\rho_R} \left(\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

$$v \frac{\partial T^*}{\partial y} + w \frac{\partial T^*}{\partial z} = \alpha_R \left(\frac{\partial^2 T^*}{\partial y^2} + \frac{\partial^2 T^*}{\partial z^2} \right)$$

Let us take : $w^* = w/U$, $z^* = z/H$,

$y^* = y/\delta$, $v^* = \dfrac{v}{\frac{\delta}{H} U}$, $\mathcal{P}^* = \dfrac{\mathcal{P}}{\rho_R U^2}$

Thus :

$$v^* \frac{\partial w^*}{\partial y^*} + w^* \frac{\partial w^*}{\partial z^*} = -\frac{\partial \mathcal{P}^*}{\partial z^*} + \frac{\beta g \Delta T H}{U^2} T^*$$
$$+ \frac{\nu_R}{U H} \frac{H^2}{\delta^2} \left(\frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\delta^2}{H^2} \frac{\partial^2 w^*}{\partial z^{*2}} \right)$$

Let's choose U & δ s.t. these two dimensionless parameters are unity.

Thus we are balancing convection diffusion & buoyancy.

$$\therefore \quad \frac{\delta}{H} = \left(\frac{\nu_R}{U H} \right)^{1/2} \qquad U = \left(\beta g \Delta T H \right)^{1/2}$$

$$\therefore \frac{\delta}{H} = \left(\frac{\nu_R}{(\beta g \Delta T H)^{1/2} H} \right)^{1/2} = \left(\frac{\nu_R^2}{\beta g \Delta T H^3} \right)^{1/4}$$

and thus :

$$v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*} = -\frac{\partial \mathcal{P}^*}{\partial z^*} + T^* + \frac{\partial^2 w^*}{\partial y^{*2}} + \left(\frac{\nu_R^2}{\beta g \Delta T\, H^3}\right)^{1/2}\frac{\partial^2 w^*}{\partial z^{*2}}$$

Provided that $\delta^2/H^2 << 1$ we may neglect the z^* diffusion term.

Likewise, we have the ~~energy~~ y-mom. eq'n:

$$\frac{\partial \mathcal{P}^*}{\partial y^*} = \frac{\delta^2}{H^2}\left\{-\left(v^*\frac{\partial v^*}{\partial y^*} + w^*\frac{\partial v^*}{\partial z^*}\right) + \frac{\partial^2 v^*}{\partial y^{*2}} + \frac{\delta^2}{H^2}\frac{\partial^2 v^*}{\partial z^{*2}}\right\}$$

$$\approx 0 \text{ if } \delta/H << 1$$

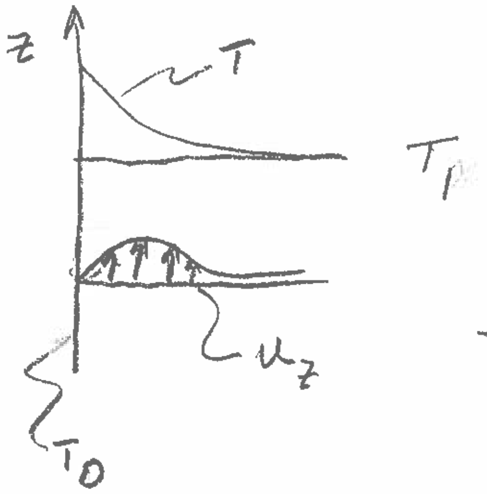
Thus the pressure <u>outside</u> the B.L. $\underline{\mathcal{P}=0}$ is impressed <u>on</u> the B.L. and

$$\underline{\underline{\frac{\partial \mathcal{P}^*}{\partial z^*} = 0}}$$

Fina

Free convection past flat plate:

Now another B.L. problem: heated plate, but now vertical w/ no forced convection



Note: since $\frac{\partial u_z}{\partial z} \neq 0$, $u_y \neq 0 \Rightarrow$ fluid is entrained in bdy layer

2-D flow, assume S.S., thus:

C.E.: $\frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0$

y-mom: $\rho\left(u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right)$

↳ not useful since $u_y << u_z$: little info. carried

Pressure distribution will be impressed on B.L.

z-mom:

$$\rho\left(u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z}\right) = -\frac{\partial P}{\partial z} - \rho g + \mu\left(\frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right)$$

Thermal E:

$$\rho \hat{C}_p\left(u_y \frac{\partial T}{\partial y} + u_z \frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

End Lec. 14

+ eqn for $\rho(T)$ (μ, k assumed cst)

Problem is Coupled & non-linear ⇒ must make some approx. to obtain sol'n.

Boussinesq's Approx.:

Let T_R be a ref temp (in this case T_1 or T_{mean}),

If fluid is at rest, N.-S. eq'ns yield

$$\underset{\sim}{\nabla} P = \rho_R \underset{\sim}{g} \quad \text{or} \quad \frac{\partial P}{\partial z} = -\rho_R g$$

$$\rho_R \longrightarrow \rho(T_R)$$

Since velocities are low, assume that pressure distrib. is given by:

$$\underset{\sim}{\nabla} P = \rho_R \underset{\sim}{g}$$ even w/ flow: Pressure in outer region impressed on B.L.

Assume μ, K, ρ_R cst except in body-force term:

$$\rho = \rho_R + \left(\frac{\partial \rho}{\partial T}\right)(T - T_R) + \dots \equiv \rho_R - \beta_R \rho_R (T - T_R)$$

$\beta_R \rightarrow$ expansion coef., $\sim \frac{1}{T_R}$ for ideal gas

∴ eq'n of motion becomes:

$$\rho_R \frac{D\underset{\sim}{u}}{Dt} = \underset{\sim}{\nabla} \cdot \underset{\approx}{\tau} \underbrace{- \rho_R \underset{\sim}{g}}_{\underset{\sim}{\nabla} P \text{ term}} + \rho \underset{\sim}{g} = \underset{\sim}{\nabla} \cdot \underset{\approx}{\tau} - \beta_R \rho_R (T - T_R) \underset{\sim}{g}$$

$$0 = \nu \frac{\partial^2 u_z}{\partial y^2} + B T^*$$

where $B \equiv \beta g (T_0 - T_1) \Rightarrow$ choose T_1 as ref.

let length scale of y be L_y, velocity scaling in z dir. be U_z:

$$\frac{\nu U_z}{L_y^2} \frac{\partial^2 u_z^*}{\partial y^{*2}} = -B T^*$$

$$\therefore \quad U_z = \left(\frac{B L_y^2}{\nu} \right)$$

Now if plate is some length H & have char. fluid velocity up it $\left(\frac{B L_y^2}{\nu}\right)$ then char. time fluid is near plate is

$$t_D = \frac{H \nu}{B L_y^2}$$

During time t_D, energy diffuses char. length:

dif$^{\underline{n}}$ length $= L_y = (\alpha t_D)^{1/2} \Rightarrow$ approp. length in y dir. thus:

$$L_y = \left(\frac{H \nu \alpha}{B L_y^2} \right)^{1/2} \quad \Xi \quad L_y = \left(\frac{H \nu \alpha}{B} \right)^{1/4}$$

Thus $y^* = \left(\frac{B}{H \nu \alpha} \right)^{1/4} y$; $u_z^* = \left(\frac{\nu}{H B \alpha} \right)^{1/2} u_z$

From C.E., obtain scaling for u_y:

$$u_y^* = \frac{u_y}{U_z}\,\frac{H}{L_y} = \left(\frac{\nu H}{\alpha^3 B}\right)^{1/4} u_y$$

Thus we obtain dimensionless eq'n:

$$\left.\frac{\partial u_y^*}{\partial y^*} + \frac{\partial u_z^*}{\partial z^*} = 0 \qquad \right\} \text{C.E.}$$

$$\left.\frac{1}{Pr}\left(u_y^* \frac{\partial}{\partial y^*} + u_z^* \frac{\partial}{\partial z^*}\right) u_z^* = \frac{\partial^2 u_z^*}{\partial y^{*2}} + T^* \right\} \text{z-mom}$$

$$\left.u_y^* \frac{\partial T^*}{\partial y^*} + u_z^* \frac{\partial T^*}{\partial z^*} = \frac{\partial^2 T^*}{\partial y^{*2}} \qquad \right\} \text{energy}$$

w/ B.C.'s: $y^*=0$: $u_z^* = u_y^* = 0$ and $T^* = 1 \rightarrow$ for $0 < z^* < 1$

$y^* = \infty$: $u_z^* = 0$ $\underline{\text{not}}$ u_y^*, $T^* = 0$

$z^* = -\infty$: $u_z^* = u_y^* = 0$, $T^* = 0$

Problem is <u>only</u> $f^n\,(Pr)$

$\therefore$ $T^*(y^*, z^*) = f^n\,(Pr)$ <u>only</u>

& for slow flow, dependence is small

What is ht. flux from plate?:

$$q_{avg} = \frac{K}{H}\int_0^H -\frac{\partial T}{\partial y}\Big|_{y=0} dz$$

$$= K(T_0-T_1)\left(\frac{B}{\nu\alpha H}\right)^{1/4}\left[\int_0^1 -\frac{\partial T^*}{\partial y^*}\Big|_{y^*=0} dz^*\right]$$

where quantity in brackets is just a weak $f^{\underline{n}}(Pr)$ $\Rightarrow$ expect to be $O(1)$

$\therefore$ let $[\quad] = C(Pr)$

from numerical sol$^{\text{n}}$s:

Pr	0.73	10	10^2	10^3
C	.517	.612	.652	.653

↑ air

we can define ht. transf. coef.:

$$h = \frac{q_{avg}}{\Delta T} = K\left(\frac{g(T_0-T_1)\beta}{\nu\alpha H}\right)^{1/4} C(Pr),$$

a rather weak $f^{\underline{n}}(T_0-T_1) \Rightarrow$ not strictly cst.

Define dimensionless group: Grashoff ※

$$Gr = \left[\frac{g\beta H^3 \Delta T}{\nu^2}\right] \equiv \frac{\text{Buoyancy forces}}{\text{Viscous forces}}$$

and Rayleigh # $Ra = Gr\,Pr$

≡ critical parameter for onset of instability, such as Rayleigh-Bénard convection in liquids htd from below: will study next wk.

In this problem:

$$q_{avg} = \frac{ck}{H}\left(T_0 - T_1\right)\left(Gr\,Pr\right)^{1/4}$$

good for laminar conv., i.e. $Gr\,Pr < 10^9$

How do we solve problem & obtain $T(z,y)$?

⇒ Semi Infinite flat plate, heated at $z=0$

Begin w/ eq'ns:

$$\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0$$

$$\left(u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z}\right) = \nu \frac{\partial^2 u_z}{\partial y^2} + BT^*$$

and $u_y \frac{\partial T^*}{\partial y} + u_z \frac{\partial T^*}{\partial z} = \alpha \frac{\partial^2 T^*}{\partial y^2}$

where we have neglected streamwise conduction, y-mom, (last inherent in Boussinesq's Approx.)

2-D, so introduce stream f$^{\underline{n}}$:

$$u_z = \frac{\partial \psi}{\partial y}, \quad u_y = -\frac{\partial \psi}{\partial z}$$

$$\therefore -\psi_z \psi_{yy} + \psi_y \psi_{yz} = \nu \psi_{yyy} + BT^* \Big\} \text{mom}$$

$$-\psi_z T^*_y + \psi_y T^*_z = \alpha T^*_{yy} \Big\} \text{energy}$$

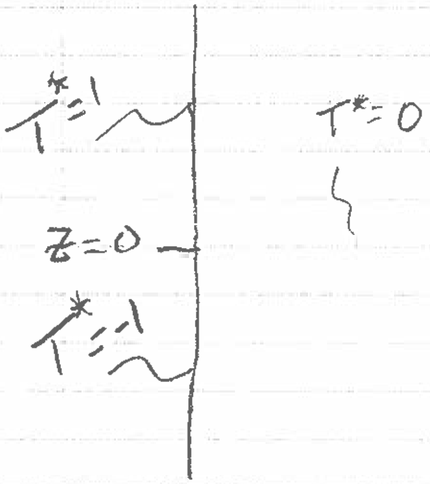
w/ B.C.'s:

$y=0$: $\psi_y = \psi_z = 0$ and $T^* = 1$

$y=\infty$: $\psi_y = T^* = 0$

and $z=0$: $\psi_y = T^* = 0$ } start w/ zero velocity at $z=0$

physically, solving slightly diff. problem:



but this gives us a homogeneous condition & makes problem solvable.

Can form ref. length & velocity from ν, α, β, so dim'l analysis will not yield sim. transf.

Try stretching:

$$T^* = a\bar{T}, \quad \psi = b\bar{\psi}, \quad y = c\bar{y}, \quad z = d\bar{z}$$

$$\therefore \frac{b^2}{c^2 d}\overline{\left(-\psi_z \psi_{yy} + \psi_y \psi_{yz}\right)} = \frac{b}{c^3}\nu \bar{\psi}_{\bar{y}\bar{y}\bar{y}} + a\beta\bar{T}$$

$$\text{or} \quad -\overline{\psi_z \psi_{yy}} + \overline{\psi_y \psi_{yz}} = \frac{d}{bc}\nu\bar{\psi}_{yyy} + \frac{c^2 d a}{b^2}\beta\bar{T}$$

$$\text{and} \quad \frac{ab}{cd}\overline{\left(-\psi_z T_y + \psi_y T_z\right)} = \frac{a}{c^2}\alpha\bar{T}_{yy}$$

$$\text{or} \quad -\overline{\psi_z T_y} + \overline{\psi_y T_z} = \frac{d}{bc}\alpha\bar{T}_{yy}$$

And from B.C.'s $a\bar{T} = 1$ at $\bar{y} = 0$

$\therefore$ invariants:

$$\left.a = 1, \quad \frac{d}{bc} = 1, \quad \frac{c^2 d a}{b^2} = 1\right\} \text{4-3=1 group of transformations!}$$

Test /wk from Friday

In canonical form, we have:

End lec. 15

$$\frac{b}{d^{3/4}}, \quad \frac{c}{d^{1/4}} \quad \text{or:} \quad \psi/z^{3/4} = f(\eta), \quad \eta = y/z^{1/4}$$

also $T^* = f''(\eta)$ <u>only</u>

Can choose dimensionless form any way we like.

Following Pollhausen (1921):

$$\zeta = \left[\frac{g\,(T_0 - T_1)}{4\,\nu^2\,T_1}\right]^{1/4} \frac{y}{z^{1/4}} \equiv \left[\frac{Gr_z}{4}\right]^{1/4} \frac{y}{z}$$

where $Gr_z \equiv \left[\dfrac{g\,\Delta T\,z^3}{\cancel{4}\,\nu^2\,T_1}\right]$

and $\psi = 4\nu\left[\dfrac{g\,\Delta T}{4\,\nu^2 T_1}\right]^{1/4} z^{3/4} f(\zeta)$

w/ $u_z = 4\nu\,z^{1/2}\left[\dfrac{g\,\Delta T}{4\,\nu^2 T_1}\right] f' = \dfrac{4\nu}{z}\left[\dfrac{Gr_z}{4}\right]^{1/2} f'$

and $u_y = \nu\left[\dfrac{g\,\Delta T}{4\,\nu^2 T_1}\right]^{1/4} \dfrac{1}{z^{1/4}}\left(\zeta f' - 3f\right)$

w/ B.C.'s: $f(0) = f'(0) = 0,\quad T^*(0) = 1$

$f'(\infty) = T^*(\infty) = 0$

↳ <u>not</u> $f(\infty) = 0$ as $\neq 0$

Plug in & get O.D.E.'s:

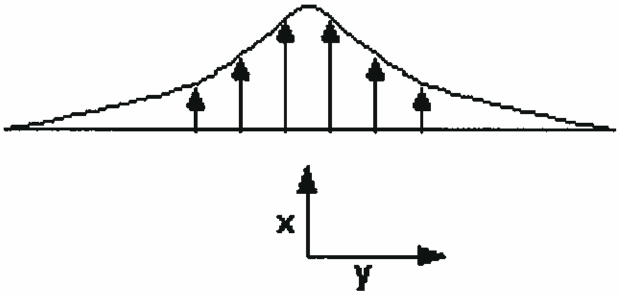
$$\left.\begin{aligned} & f''' + 3ff'' - 2(f')^2 + T^* = 0 \\ & T^{*\prime\prime} + 3\,Pr\,T^{*\prime} f = 0 \end{aligned}\right\}$$

still $f''(Pr) \Rightarrow$ shifted from mom. to E by choice of ψ^*, ζ

From Senior Lab

Flow due to a Heated Wire: The Theoretical Flow Field

In the vicinity of a line source of energy (in this case the source is a wire carrying electrical current) the temperature is elevated. If the fluid surrounding the source expands upon heating (as is the case for most fluids) the fluid near the wire will have a density less than that of the fluid far from the wire and hence will rise, driving a natural convection circulation pattern. The velocity field produced is approximately as depicted below:



Note that we take the x direction to be *vertical* and the y direction to be horizontal (the z-direction is along the wire and does not figure into a 2-D problem). This is because the flow field may be described as a *boundary layer* (much the same as flow past a flat plate studied in fluids last year), and x is usually taken to be the distance along the boundary. While there isn't a boundary here, there *is* a separation of length scales: the length scale in the horizontal (y) direction is much shorter than the length scale in the vertical (x) direction (as we shall demonstrate), which is really what is required for a boundary layer type flow.

This *Thermal Plume* boundary layer problem is described in a number of transport texts. One reference is L. G. Leal, *Laminar Flow and Convective Transport Processes*, 1992, p. 691. This is the text currently used in the spring semester graduate fluid mechanics course. A complete review of the theory and experiment associated with this problem is provided by Gebhart, B., *et al., Buoyancy-Induced Flow and Transport*, 1988. Both texts have been placed on reserve in the Engineering library.

To derive the equations governing the flow field we need to make a number of assumptions or approximations, which ultimately limit the validity of our solution. We begin with the Boussinesq approximation: we assume that the density fluctuations resulting from the temperature variations are sufficiently small that the only place where they matter is in the buoyancy term of the equations of motion. This is usually a very good approximation for natural convection flows. We also take all other material properties (viscosity, thermal diffusivity, etc.) to be constant (not as good an approximation!). We assume the flow to be strictly 2-D with no variation in the z-direction. This would be correct for steady-state flows with an infinitely long wire: obviously we'll get into trouble if we are too far above a wire of finite length, such that it starts to look like a point source! Finally, we shall ignore all effects of the boundaries - both the side walls (they will induce a finite amplitude recirculation flow) and the top and bottom. Again, this won't be too bad provided that the flow profile is very narrow, and if we are far away from the boundaries!

We will use conservation of mass, momentum, and energy to describe the flow field mathematically. Thus:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\rho g$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$

where α is the coefficient of thermal diffusivity and μ is the viscosity (assumed constant). The density profile is given by:

$$\rho=\rho_0\left(1-\beta\left(T-T_0\right)\right)$$

where β is the coefficient of thermal expansion ($[\Delta V/V]/\Delta T$) and ρ_0 and T_0 are the density and temperature far from the source, respectively.

Far from the wire the temperature and density are constants, and the velocity in the x-direction due to the wire is zero. In this region the pressure distribution is simply that due to hydrostatic pressure variation $p_\infty = p_0 - \rho_0\, g\, x$. If we subtract this off (e.g., let $p = P + p_\infty$), then the momentum equations become:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\beta g\left(T-T_0\right)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

where ν is the kinematic viscosity μ/ρ, and we have divided through by the density.

We also have the boundary conditions:

$$u\Big|_{y\to\pm\infty}\to 0$$

$$P\Big|_{y\to\pm\infty}\to 0$$

$$T\Big|_{y\to\pm\infty}\to T_0$$

$$\int_{-\infty}^{+\infty}\left[\rho\,\hat{C}_p\,u\left(T-T_0\right)-k\frac{\partial T}{\partial x}\right]dy=Q/L$$

where Q/L is the energy dissipated per unit length of the wire. This last condition is a statement of conservation of energy. The energy produced by the electrical dissipation has to go somewhere: at steady state the total integrated energy flux (both convection and conduction) through any plane of constant x must equal the energy released by the wire.

To proceed further we need to render the equations dimensionless. Unlike forced convection problems such as flow past a flat plate, we don't have a reference velocity, temperature, or even a length scale imposed by the boundary conditions. In this case what we shall do is take these quantities to be unknown constants, plug them into the equations, and then choose them so that all the important terms are scaled to be O(1). Thus we take:

$$x^* = \frac{x}{L_x} \ ; \ y^* = \frac{y}{L_y} \ ; \ u^* = \frac{u}{U_c} \ ; \ v^* = \frac{v}{V_c} \ ; \ P^* = \frac{P}{P_c} \ ; \ T^* = \frac{T - T_0}{T_c}$$

We begin our scaling with the continuity equation. Substituting in and dividing through to render the equation dimensionless yields:

$$\frac{\partial u^*}{\partial x^*} + \left[\frac{V_c L_x}{U_c L_y}\right] \frac{\partial v^*}{\partial y^*} = 0$$

It is *always* the case that, when scaling two dimensional flow problems, both terms of the continuity equation must be of the same magnitude. Thus we choose the quantity in brackets to be unity, and we get:

$$V_c = U_c \left[\frac{L_y}{L_x}\right]$$

e.g., the characteristic velocity in the y-direction is related to that in the x-direction by the ratio of the two length scales. Since we are looking for a boundary-layer type solution, we expect this ratio to be much less than unity.

We now turn to the x-momentum equation. If we substitute back in for V_c, we obtain:

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\left[\frac{P_c}{\rho\, U_c^2}\right] \frac{\partial P^*}{\partial x^*} + \left[\frac{\nu L_x}{U_c L_y^2}\right] \left(\frac{\partial^2 u^*}{\partial y^{*2}} + \left[\frac{L_y^2}{L_x^2}\right] \frac{\partial^2 u^*}{\partial x^{*2}} \right) + \left[\frac{\beta\, g\, T_c L_x}{U_c^2}\right] T^*$$

The physical mechanisms which must be preserved are the buoyancy source term (that's what drives the entire flow), diffusion of momentum in the y-direction, and the momentum convection term. We will demonstrate in a bit that the pressure gradient term is negligible for natural convection flows. Thus we shall take:

$$\left[\frac{\beta\, g\, T_c L_x}{U_c^2}\right] = 1 \ \text{ and } \ \left[\frac{\nu L_x}{U_c L_y^2}\right] = 1$$

which yields two relations between U_c, T_c, L_x, and L_y. The physical interpretation of the boundary layer length scale (the second relation) is very simple. The time necessary for fluid to be convected in the x direction a distance L_x with velocity U_c is just $t \sim L_x/U_c$. During this time momentum diffuses in the y-direction a distance:

$$L_y = (\nu\, t)^{1/2} = \left(\frac{L_x \nu}{U_c}\right)^{1/2}$$

which is the appropriate length scale in the y-direction, at least for the momentum boundary layer. Note that diffusion in the x-direction is of order $[L_y^2/L_x^2]$ relative to the other terms in the equation. Terms of this order can be neglected in the boundary layer limit, e.g., when $[L_y^2/L_x^2] << 1$.

Next up in our scaling is the y-momentum equation, which we shall use to obtain the correct scaling for pressure variations due to flow. Again substituting in and using the scalings for L_y and V_c derived above we obtain after some manipulation:

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\left[\frac{P_c L_x}{\mu U_c}\right] \frac{\partial P^*}{\partial y^*} + \frac{\partial^2 v^*}{\partial y^{*2}} + \left[\frac{L_y^2}{L_x^2}\right] \frac{\partial^2 v^*}{\partial x^{*2}}$$

From this equation, we find the appropriate scaling for the pressure variation in the boundary layer is $P_c = \mu\, U_c / L_x$. We can use this in the x-momentum equation together with the definition of L_y to demonstrate that the pressure gradient term is negligible, e.g.,

$$\left[\frac{P_c}{\rho U_c^2}\right] = \left[\frac{P_c}{\rho U_c^2}\right]\left[\frac{\mu U_c}{P_c L_x}\right] = \left[\frac{L_y^2}{L_x^2}\right]\left[\frac{\nu L_x}{U_c L_y^2}\right] = \left[\frac{L_y^2}{L_x^2}\right] << 1$$

Now we look at the energy equation, where we have made the V_c substitution and divided through:

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \left[\frac{\alpha L_x}{U_c L_y^2}\right]\left(\frac{\partial^2 T^*}{\partial y^{*2}} + \left[\frac{L_y^2}{L_x^2}\right]\frac{\partial^2 T^*}{\partial x^{*2}}\right)$$

We have already scaled L_y with U_c and L_x in the momentum equation, however. The scaled energy equation thus becomes:

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{\mathrm{Pr}}\left(\frac{\partial^2 T^*}{\partial y^{*2}} + \left[\frac{L_y^2}{L_x^2}\right]\frac{\partial^2 T^*}{\partial x^{*2}}\right)$$

where diffusion in the x-direction is again of order $[L_y^2/L_x^2]$ relative to the other terms in the equation and thus is neglected. It is interesting to note that the Prandtl number $\mathrm{Pr} = \nu/\alpha$ divides the diffusive term in the energy equation. We can't get rid of this ratio by scaling, since it is a material property of the fluid. It always appears in either the momentum or energy equations depending on whether we balance the convective or diffusion terms in the momentum or energy equations. For problems where Pr >> 1, such as is the case in this experiment, the thermal boundary layer will be much thinner than the momentum boundary layer. In our scaling the momentum boundary layer (e.g., the velocity profile we are trying to measure!) will be O(1) in thickness, while the thermal boundary layer will be $O(1/\mathrm{Pr}^{1/2})$ in thickness. You can actually see the thermal boundary layer as a very thin shimmering sheet due to refractive index variations if you look closely (laser off, please!) at the profile. Note that you can't just throw out the diffusive term for large Pr, as you would not then be able to satisfy the boundary conditions - instead it just leads to a thin thermal boundary layer within the thicker momentum boundary layer! Some references actually rescale y* in the energy equation to account for this difference in widths, however we shall not do so here.

The final bit of scaling comes from the integral boundary condition. Again inserting the above scalings and dividing through yields:

$$\int_{-\infty}^{+\infty}\left(u^*\, T^* - \left[\frac{L_y^2}{L_x^2}\right]\frac{\partial T^*}{\partial x^*}\right)dy^* = \left[\frac{Q/L}{\rho\, \hat{C}_p U_c\, T_c\, L_y}\right]$$

Again, we take the quantity in brackets (on the right hand side) to be unity, and ignore the x-diffusion term which is of order [L_y^2/L_x^2]. Putting all this together we get three equations for U_c, T_c, and L_y. in terms of L_x:

$$\left[\frac{\beta\, g\, T_c\, L_x}{U_c^2}\right] = 1 \;\; ; \;\; \left[\frac{\nu\, L_x}{U_c\, L_y^2}\right] = 1 \;\; ; \;\; \left[\frac{Q/L}{\rho\, \hat{C}_p\, U_c\, T_c\, L_y}\right] = 1$$

After some manipulation, these can be solved to yield:

$$L_y = \left[\frac{\nu^3 \rho\, \hat{C}_p}{(Q/L)\, \beta\, g}\, L_x^2\right]^{1/5} \;\; ; \;\; U_c = \left[\left(\frac{(Q/L)\,\beta\, g}{\rho\, \hat{C}_p}\right)^2 \frac{L_x}{\nu}\right]^{1/5} \;\; ; T_c = \left[\frac{(Q/L)^4}{\nu^2\, \beta\, g\, (\rho\, \hat{C}_p)^4\, L_x^3}\right]^{1/5}$$

and the ratio:

$$\left(\frac{L_y}{L_x}\right)^2 = \left[\frac{\nu^3\, \rho\, \hat{C}_p}{(Q/L)\,\beta\, g}\, L_x^{-3}\right]^{2/5}$$

It is interesting to note that if the scaling of the equations is done correctly, all the dimensionless -variables- (e.g., the *'ed quantities) should be of O(1). In fact, when the equations are solved numerically, and L_x is chosen to be the distance above the wire, the dimensionless vertical velocity at the centerline of the flow has a value of about 0.9 for Pr >> 1, comfortingly O(1). The width of the profile should also be of O(1), with the numerical solution yielding a value of around 1.2 for where the velocity falls to 50% of its maximum. This O(1) property of scaling means that you can make a pretty good guess of the magnitude of the velocity and profile width without ever actually solving the equations! It also suggests that if the dimensionless values you get by solving the equations -aren't- of order unity, then you should probably go back and re-scale things!

While the characteristic scaling values derived above look complex, they arose in a natural way from the balancing of physical mechanisms in the equations of motion. The dimensionless equations, neglecting all terms which are of O([L_y^2/L_x^2]), become:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$u^*\frac{\partial T^*}{\partial x^*} + v^*\frac{\partial T^*}{\partial y^*} = \frac{1}{\Pr}\frac{\partial^2 T^*}{\partial y^{*2}}$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{\partial^2 u^*}{\partial y^{*2}} + T^*$$

$$\int_{-\infty}^{+\infty} u^* T^* dy^* = 1$$

In the above derivation we have made a number of approximations to simplify the equations. In no particular order, these are:

1) The Boussinesq approximation, $\beta T_c << 1$; $\frac{1}{\mu}\frac{\partial \mu}{\partial T} T_c << 1$ (no viscosity variation)

2) Two-Dimensional flow, $[L_y^2/L_z^2] << 1$ (L_z is the length of the wire)

3) Steady-state flow

4) The boundary layer approximation, $[L_y^2/L_x^2] << 1$.

5) An infinite medium (no wall effects)

It is interesting to note that all parameters except the Prandtl number have been scaled out of these equations, and yet we have not had to specify L_x, the characteristic length scale in the x-direction. This occurs often in boundary-layer problems (problems where there is a large separation in the length scales in flow and cross-streamline directions), and results in a similarity solution.

A key result of the above scaling analysis is the dependence of the characteristic length scale in the y-direction and the characteristic velocity on L_x. We find that U_c varies as $L_x^{1/5}$, and L_y varies as $L_x^{2/5}$. This suggests that the velocity will increase as $x^{1/5}$ power as we move up above the wire, and that the width of the velocity profile will increase as $x^{2/5}$. We will make use of these scalings in developing our similarity solution to the transport equations.

The flow pattern produced by a line source is two dimensional, thus we may define a stream function ψ:

$$u^* = \frac{\partial \psi^*}{\partial y^*} \; ; \; v^* = -\frac{\partial \psi^*}{\partial x^*}$$

In terms of the streamfunction the equations become:

$$\psi_y^* \psi_{xy}^* + \psi_x^* \psi_{yy}^* = \psi_{yyy}^* + T^*$$

$$\psi_y^* T_x^* + \psi_x^* T_y^* = \frac{1}{\Pr} T_{yy}^* \; ; \; \int_{-\infty}^{+\infty} \psi_y^* T^* dy^* = 1$$

where subscripts denote partial derivatives with respect to the argument. The similarity transform is given by:

$$\psi^* = x^{*3/5} f(\eta) \; ; \; T^* = x^{*-3/5} g(\eta) \; ; \; \eta = \frac{y^*}{x^{*2/5}}$$

$$\text{with } u^* = \psi_y^* = x^{*1/5} f'(\eta)$$

$$\text{and } v^* = -\psi_x^* = -x^{*-2/5}\left(\frac{3}{5} f(\eta) - \frac{2}{5} \eta f'(\eta)\right)$$

We may also find the velocity derivative of interest:

$$\frac{\partial v^*}{\partial y^*} = -\psi^*_{xy} = -x^{*-4/5}\left(\frac{1}{5}f'(\eta) - \frac{2}{5}\eta f''(\eta)\right)$$

and the resulting ODE's:

$$f''' = -g + \left[\frac{1}{5}(f')^2 - \frac{3}{5}f f''\right] \; ; \; g' = -\mathrm{Pr}\,\frac{3}{5}f g$$

where primes denote derivatives with respect to the similarity variable η. We also have the transformed boundary conditions:

$$f(0) = f''(0) = 0$$

$$f' \rightarrow 0 \text{ as } \eta \rightarrow \infty$$

$$\int_{-\infty}^{+\infty} f' g \, d\eta = 1$$

The boundary conditions at η = 0 are symmetry conditions: both v* and $\frac{\partial u^*}{\partial y^*}$ must vanish at the centerline. These equations may be solved numerically as a function of the Prandtl number. The procedure for solving the equations numerically is to set them up as a system of four first order differential equations for f, f ', f '', and g. We have initial conditions for f and f '', but we must determine initial conditions for f ' (the dimensionless velocity) and g (the dimensionless temperature) at the centerline. We do this by using the shooting method, guessing values of f '(0) and g(0) and iterating until both the boundary condition f '(∞) = 0 and the integral boundary condition are satisfied. The integral boundary condition may be tracked by simply adding its integral as a fifth first order differential equation in the integration process.

Plots of the solution for Pr =68 are given in the following diagram. Note that the dimensionless velocities have been scaled by the dependence on x* predicted by the similarity solution. The program which generated this diagram is available at the URL:

http://www.nd.edu/~dtl/cheg459/pivexperiment

You should be prepared to discuss these theoretical results and, in particular, be able to estimate where the approximations which led to the solution will break down.

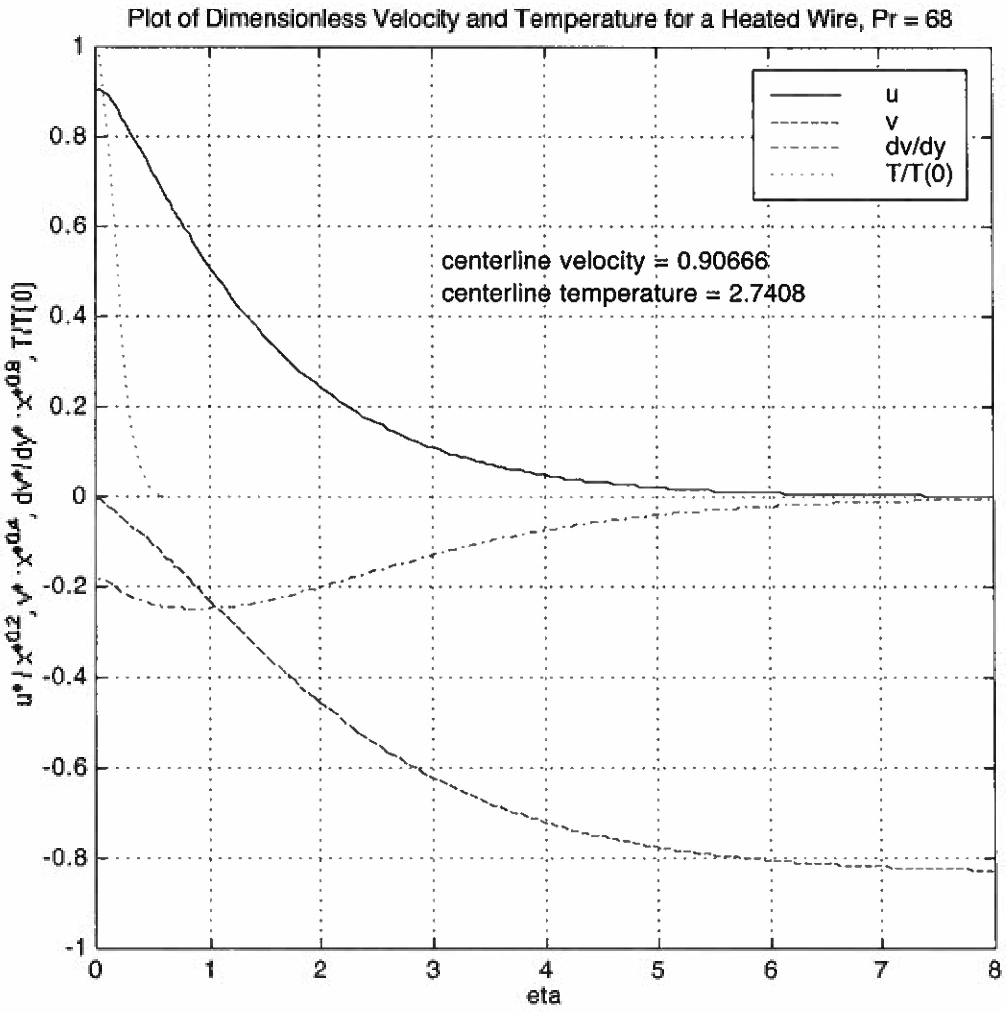
The following fluid properties may be useful:

50% by volume glycerin / water:

$\nu = 7.46\text{x}10^{-2}$ cm^2/s
$C_p = 3.09$ J/g^{o}C
$\rho = 1.145$ g/cm^3
$\alpha = 1.11\text{x}10^{-3}$ cm^2/s
$\beta = 5.3\text{x}10^{-4}$ 1/^{o}C
$n_D = 1.407$

Properties of glycerin-water solutions as a function of concentration may be obtained from the Dow Chemical website:

http://www.dow.com/glycerine/resources/physicalprop.htm



Thus have 5th Order non-linear ODE w/ 5 B.C.'s may solve numerically, no known exact sol'n. Still, a vast simplification

BS&V 10.6

Have now examined forced & free conv. probs. in detail, now step back & conduct generalized dimensional analysis of eq'n of change:

For cst. properties (except $\rho(T)$ for Nat. Conv.) obtain eq'ns:

$$\text{C.E.}: \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{u} = \begin{cases} 0 & \text{incompressible} \\ -\frac{1}{\rho}\frac{D\rho}{Dt} & \text{compressible} \end{cases}$$

$$\text{N.S.}: \quad \rho \frac{D\underset{\sim}{u}}{Dt} = \mu \nabla^2 \underset{\sim}{u} + \begin{cases} -\nabla P + \rho \underset{\sim}{g} & \text{forced} \\ -\rho\beta \underset{\sim}{g}\,(T-T_0) & \text{free} \end{cases}$$

$$\text{Energy}: \quad \rho \hat{C}_p \frac{DT}{Dt} = \underbrace{k\nabla^2 T}_{\text{cond.}} + \underbrace{\mu\, \phi_v}_{\text{irrev. visc. dissip}} + \underbrace{\beta T \frac{DP}{Dt}}_{\text{compression (rev. work)}}$$

Suppose we have some general forced convection problem, assoc. length scale D, 2 temps T_1 & T_0, fluid velocity U, ref. pressure P_0:

$$\underset{\sim}{x}^* = \underset{\sim}{x}/D, \quad \underset{\sim}{u}^* = \underset{\sim}{u}/U, \quad T^* = \frac{T-T_0}{T_1-T_0}$$

$$p^* = \frac{P-P_0}{\rho U^2}, \quad t^* = \frac{tU}{D}$$

Not always correct scaling: some cases no refi length, or velocity (shear flow), etc. => often in these cases get sim. transform.

Obtain Dimensionless eq^{n}s:

$$\underset{\sim}{\nabla}^* \cdot \underset{\sim}{u}^* = 0 \quad \text{(incompressible)}$$

$$\frac{D\underset{\sim}{u}^*}{Dt^*} \equiv \frac{\partial \underset{\sim}{u}^*}{\partial t^*} + \underset{\sim}{u}^* \cdot \underset{\sim}{\nabla}^* \underset{\sim}{u}^* = -\underset{\sim}{\nabla}^* p^* + \frac{1}{Fr}\frac{\underset{\sim}{g}}{g} + \frac{1}{Re}\nabla^{*2}\underset{\sim}{u}^*$$

where $Fr = \frac{U^2}{Dg} \equiv \frac{\text{inertial forces}}{\text{gravitational forces}}$ & $Re = \frac{\rho U D}{\mu}$

and $$\frac{DT^*}{Dt^*} = \frac{1}{Re\,Pr}\nabla^{*2}T^* + \frac{Br}{Re\,Pr}\phi_v^*$$

$$+ \frac{Br}{Pr}\left[\beta T_0 + \beta(T_1 - T_0)\,T^*\right]\frac{DP^*}{Dt^*}$$

where $Br = \text{Brinkman } \# \equiv \frac{\mu U^2}{K(T_1 - T_0)} = \frac{\text{viscous dissip}}{\text{conduction}}$

$Pr = \frac{\nu}{\alpha} = \frac{\text{mom. diffusivity}}{\text{thermal diffusivity}}$

Can also define Eckert # $= Ec = \frac{\rho U^2}{\rho C_p (T_1 - T_0)} \equiv \frac{\text{inertial } E}{\text{Thermal } E}$

$$Ec = \frac{Br}{Pr}$$

Thus, formally,

$$\rho^* = p^*\left[t^*, \underset{\sim}{x}^*; Re, Pr, Fr, Br, \beta T_0, \beta(T_r - T_0)\right]$$

w/ sim. relation for $\underset{\sim}{u}^*$ & T^*

May get add'l groups from B.C.'s, Ex:

e.g. periodic temp. profile, some frequency f

obtain group $Sr = \frac{fD}{U} \equiv$ strouhal #,

thus $T^* = f^{\underline{n}}(Sr)$ as well

Dynamic similarity is preserved if all dim. groups are identical between two dif. physical probs. ⇒

Used in scale up of systems.

For free convection:

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0$$

$$\rho \frac{D\underset{\sim}{u}}{Dt} = \mu \nabla^2 \underset{\sim}{u} - \rho \beta \underset{\sim}{g} (T - T_0)$$

$$\rho \hat{C}_p \frac{DT}{Dt} = k \nabla^2 T + \mu \phi_v + \beta T \frac{DP}{Dt}$$

Use D as length scale

No ref. velocity, so choose ν/D

$$\therefore \; \underset{\sim}{x}^* = x/D \; ; \; \underset{\sim}{u}^* = \frac{D}{\nu} \underset{\sim}{u} \; ; \; P^* = \frac{P - P_0}{\rho \nu^2 / D^2}$$

$$t^* = \frac{\nu t}{D^2} \; ; \; T^* = \frac{T - T_0}{T_1 - T_0}$$

Obtain eq^ns:

$$\underset{\sim}{\nabla}^{*}\cdot\underset{\sim}{u}^{*}=0$$

$$\frac{D\underset{\sim}{u}^{*}}{Dt^{*}}=\nabla^{*2}\underset{\sim}{u}^{*}-Gr\,\frac{\underset{\sim}{g}}{g}\,T^{*}$$

$$\frac{DT^{*}}{Dt^{*}}=\frac{1}{Pr}\nabla^{*2}T^{*}+\left[\frac{\nu^{2}}{D^{2}\hat{C}_{p}\Delta T}\right]\phi_{v}^{*}$$

$$+\left[\frac{\nu^{2}}{D^{2}\hat{C}_{p}\Delta T}\right]\left[\beta T_{0}+\beta\Delta T\,T^{*}\right]\frac{DP^{*}}{Dt^{*}}$$

where $\left[\frac{\nu^{2}}{D^{2}\hat{C}_{p}\Delta T}\right]=Ec$

and $Gr=\frac{g\beta\Delta T D^{3}}{\nu^{2}}\equiv\frac{\text{Buoyancy forces}}{\text{viscous forces}}$

for free convection:

$$T^{*}=f^{n}\left(t^{*},\underset{\sim}{x}^{*};Gr,Pr,\underline{Ec},\beta T_{0},\beta\Delta T\right)$$

small dep. since compressibility, viscous dissip. negligible

Free & Forced Convection:

General problem where both are present

Define coord. s.t. $\underset{\sim}{g}$ aligned w/ y axis:

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0$$

$$\frac{Du_x}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u_x$$

$$\frac{Du_y}{Dt} = g\beta(T - T_0) + \nu \nabla^2 u_y$$

→ no applied pressure gradient in vertical direction!
↳ otherwise forced conv. would swamp.

$$\frac{Du_z}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 u_z$$

let $\underset{\sim}{x}^* = \underset{\sim}{x}/D$, $\underset{\sim}{u} = \underset{\sim}{u}/U$, $T^* = \dfrac{T - T_0}{T_1 - T_0}$

$$p^* = \frac{p - p_0}{\rho U^2} \qquad t^* = \frac{tU}{D}$$

Thus:

$$\underset{\sim}{\nabla}^* \cdot \underset{\sim}{u}^* = 0$$

$$\frac{Du_x^*}{Dt^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re}\nabla^{*2} u_x^*$$

$$\frac{Du_y^*}{Dt^*} = \left(\frac{Gr}{Re^2}\right) T^* + \frac{1}{Re}\nabla^{*2} u_y^*$$

$$\frac{Du_z^*}{Dt^*} = -\frac{\partial p^*}{\partial z^*} + \frac{1}{Re}\nabla^{*2} u_z^*$$

+ energy eq²n:

$$\frac{DT^*}{Dt^*} = \left(\frac{1}{Re\,Pr}\right)\nabla^{*2}T^* + \left(\frac{Br}{Re\,Pr}\right)\phi_v^* + \left(\frac{Br}{Pr}\right)\left[\beta T_0 + \beta\Delta T\,T^*\right]\frac{Dp^*}{Dt^*}$$

where $Re = \frac{D\rho U}{\mu}$, $Pr = \frac{\hat{C}_p\mu}{K}$

$$Gr = \frac{g\beta(T_1-T_0)D^3}{\nu^2}, \quad Br = \frac{\mu U^2}{K(T_1-T_0)}$$

and $T^* = f^{\Delta}(t^*, \underset{\sim}{x}^*; Re, Pr, Gr, Br, \beta T_0, \beta\Delta T)$

$$Nu_{ave} = \frac{hD}{K} = \frac{Q}{A\,\Delta T}\;\frac{D}{K}$$

$$Q = \int_{\partial D} -K\frac{\partial T}{\partial \underset{\sim}{x}}\cdot n\,dA = -DK(T_1-T_0)\int_{\partial D^*}\frac{\partial T^*}{\partial \underset{\sim}{x}^*}\cdot \underset{\sim}{n}\,dA^*$$

$$\left. \right\rbrace dA^* = dA/D^2$$

$$\left. \right\rbrace \underset{\sim}{x}^* = \underset{\sim}{x}/D$$

$$\therefore Nu_{ave} = -\int_{\partial D^*}\frac{\partial T^*}{\partial \underset{\sim}{x}^*}\cdot \underset{\sim}{n}\,dA^* = f^{\underline{u}}\left[t^*, \text{geometry}, Re, Pr, Gr, Br, \beta T_0, \beta\Delta T\right]$$

Let's look at importance of some terms

If we have forced convection, what is sig. of free convection?

Air flow, $U \sim 1 \, m/sec$ (about 2 mph ⇒ slow)

let $\Delta T = 50°C$, $D = 30 \, cm$ (1 ft), $\beta = \frac{1}{300°K}$

Air properties: $\rho = 1.2 \, kg/m^3$, $\nu = 15 \times 10^{-6} \, m^2/sec$

$\hat{C}_p = 1000 \, J/kg \, °K$, $k = 0.024 \, J/s \cdot m \cdot °K$

$$Pr = 0.73 = \frac{\nu}{\alpha}$$

$$Gr = \frac{g \beta \Delta T D^3}{\nu^2} = 2 \times 10^8$$

$$Re = \frac{DU}{\nu} = 2 \times 10^4$$

$\therefore \frac{Gr}{Re^2} = 0.5 \Rightarrow O(1)$ so can't neglect buoyancy terms.

and $Br = 1.5 \times 10^{-5}$ so viscous heating, compression effects are neglig. ble.

Now for ~~gentle~~ stiff breeze

(~22 mph)

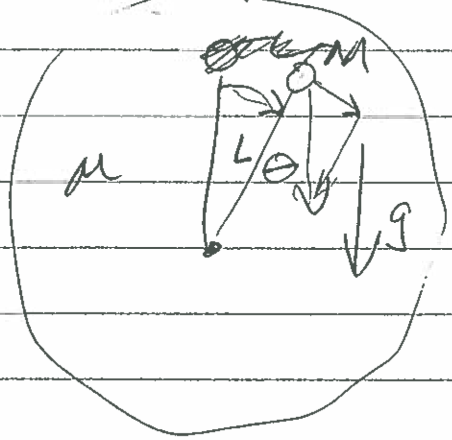
$U \sim 10 \, m/sec \therefore Re = 2 \times 10^5$

$\therefore \frac{Gr}{Re^2} = .005$ & can neglect buoyancy rel. to forced convection.

End lec. 16

Linear Stability

~~Sub~~ Damped Pendulum



$$ML\frac{d^2\theta}{dt^2} = -6\pi\mu a L\frac{d\theta}{dt} + Mg\sin\theta$$

We seek equilibrium (steady) sol'ns:

$\sin\theta = 0 \therefore \theta = 0, \pi$ (base states)

let $\theta_0 = \pi$, take $\theta_0 = \pi + \theta'(t)$

$$\therefore ML\frac{d^2\theta'}{dt^2} = -6\pi\mu a L\frac{d\theta'}{dt} - Mg\sin\theta'$$

linearization

$$= -6\pi\mu a L\frac{d\theta'}{dt} - Mg\theta'$$

let $\theta' = Ae^{st}$

$$\therefore AMLs^2e^{st} = -6\pi\mu a L\, s\, Ae^{st} - Mg\, Ae^{st}$$

or $MLs^2 = -6\pi\mu a L\, s - Mg$

$$s^2 + \frac{6\pi\mu a}{M}s + \frac{g}{L} = 0$$

$$S = \frac{-\frac{6\pi\mu a}{M} \pm \sqrt{\left(\frac{6\pi\mu a}{M}\right)^2 - 4\frac{g}{L}}}{2}$$

where $Re(S) < 0$ for all values above

Note that $Im(s) \neq 0$ for $\left(\frac{6\pi\mu a}{M}\right)^2 < 4\frac{g}{L}$

Which indicates a periodic part to the solution.

For base state $\theta_0 = 0$ we have:

$$ML\frac{d^2\theta'}{dt^2} = -6\pi\mu a L\frac{d\theta'}{dt} + Mg\theta'$$

which yields:

$$s^2 + \frac{6\pi\mu a}{M}s - \frac{g}{L} = 0$$

so:

$$s = \frac{-\frac{6\pi\mu a}{M} \pm \sqrt{\left(\frac{6\pi\mu a}{M}\right)^2 + 4\frac{g}{L}}}{2}$$

which has a positive root for all g/L, so is unstable (and purely real)

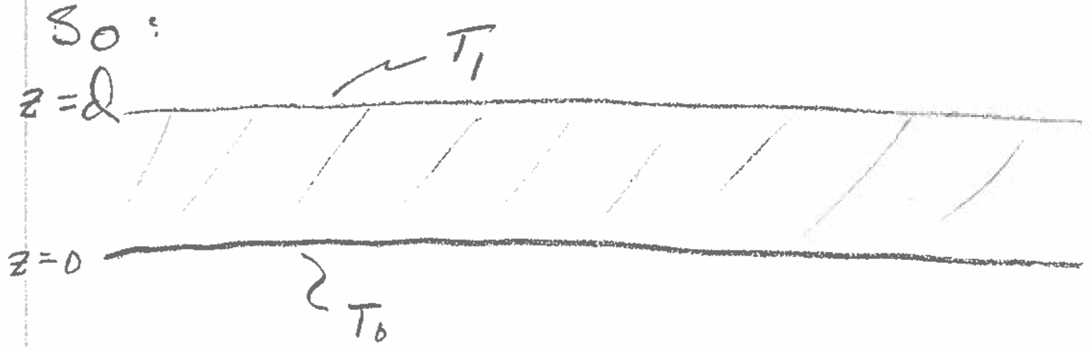
Rayleigh-Bénard convection

Ref: Drazin & Reid, Hydrodynamic Stability Ch. 2, Chandresekhar: Hydrodynamic & Hydromagnetic Stability

1900: Bénard described liquid heated from below, found it formed hexagonal (4-7 sides) cells ⇒ instability

1916: Rayleigh solved problem, showed instability occurred only if adverse temp. gradient exceeded critical value, named Rayleigh # in his honor

So:

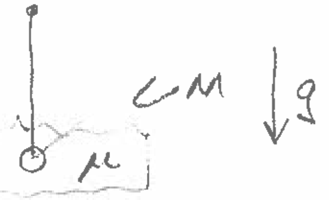


fluid is at rest between inf. planes

We need to calculate base state & then det. under what conditions base state will be unstable to small perturbations:

Unstable if we perturb & fluctuations grow w/ time:

Damped pendulum:



Base state: mass hanging down, sol'n to steady eq'ns. Stable sol'n ⇒ perturb. decay

If mass pointing up, also sol'n to steady eq'ns, but unstable ⇒ perturb. grow.

Base State for R-B prob:

$$\underset{\sim}{U} = 0, \quad T^b = T_0 - \frac{T_0 - T_1}{d} z \quad , \text{ now calc. } P(z)$$

$$p^b = P_0 - g\rho_0\left(z + \frac{1}{2}\beta\frac{T_0 - T_1}{d}z^2\right) \quad 0 \le z \le d$$

P_0 ↳ p at z = 0.

↑ integral of $\rho(z)$ using Boussinesq's Approx.

↳ integral of $\frac{\partial p}{\partial z} = -\rho g$

$$\rho = \rho_0\left\{1 - \beta(T - T_0)\right\}$$

Eq'ns governing motion:

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0$$

$$\frac{D\underset{\sim}{u}}{Dt} = -\underset{\sim}{\nabla}\left(\frac{p}{\rho_0} - \underset{\sim}{g}\cdot\underset{\sim}{X}\right) - \beta\underset{\sim}{g}\left(T - T_0\right) + \nu\nabla^2\underset{\sim}{u}$$

$$\frac{DT}{Dt} = \alpha\nabla^2 T$$

Now let

$\underset{\sim}{u} = \underset{\sim}{U}^b + \underset{\sim}{u}'(\underset{\sim}{x}, t)$; $T = T^b + T'(\underset{\sim}{x}, t)$;

$p = P^b + p'(\underset{\sim}{x}, t)$

where perturbations from base state are <u>small</u> => neglect non-linear terms

$$\therefore \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{u}' = 0 \quad \Big\} \text{ C. E.}$$

Energy eq'n:

$$\frac{\partial T}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} T = \alpha \nabla^2 T$$

$$\therefore \quad \frac{\partial T^b}{\partial t} + \frac{\partial T'}{\partial t} + \underset{\sim}{u}' \cdot \underset{\sim}{\nabla} T^b + \underset{\sim}{U}^b \cdot \underset{\sim}{\nabla} T' + \underset{\sim}{U}^b \cdot \underset{\sim}{\nabla} T^b + \underset{\sim}{u}' \cdot \underset{\sim}{\nabla} T' =$$

$$\alpha \nabla^2 T^b + \alpha \nabla^2 T'$$

$$\equiv \left[\frac{\partial T^b}{\partial t} + \underset{\sim}{U}^b \cdot \underset{\sim}{\nabla} T^b - \alpha \nabla^2 T^b \right] \rightsquigarrow = 0!$$

=> base state sol'n

$$+ \left[\underset{\sim}{u}' \cdot \underset{\sim}{\nabla} T^b + \underset{\sim}{U}^b \cdot \underset{\sim}{\nabla} T' - \alpha \nabla^2 T' \right] = - \underset{\sim}{u}' \cdot \underset{\sim}{\nabla} T'$$

small, as $O(\varepsilon^2)$ => lineari

Also, $\underset{\sim}{U}^b = 0$

$$\therefore \quad \frac{\partial T'}{\partial t} + \underset{\sim}{u}' \cdot \underset{\sim}{\nabla} T^b = \alpha \nabla^2 T' \quad \Big\} \text{ linearized eq'n of E}$$