Similarly, from N.-S. eq'ns:

$$\frac{\partial \underset{\sim}{u}'}{\partial t} = -\underset{\sim}{\nabla}\left(\frac{p'}{\rho_0}\right) - \beta \underset{\sim}{g} T' + \nu \nabla^2 \underset{\sim}{u}' \Big\} \text{motion}$$

Now render dimensionless. choose $d$ as length, $\frac{d^2}{\alpha}$ as time, $T_0 - T_1$ as $\Delta T$

$$\therefore \underset{\sim}{x}^* = \underset{\sim}{x}/d, \quad t^* = \frac{\alpha t}{d^2}, \quad \underset{\sim}{u}^* = \underset{\sim}{u}' \frac{d}{\alpha}, \quad T^* = \frac{T - T_0}{T_0 - T_1}$$

$$\therefore T^{b*} = -z^*, \quad \longrightarrow p^* = (p - p_0)\frac{d^2}{\rho_0 \alpha^2}$$

and we obtain the dimensionless eq'ns:

C.E.: $\underset{\sim}{\nabla}^* \cdot \underset{\sim}{u}^{*\prime} = 0$

$$\frac{\partial T^{*\prime}}{\partial t^*} - \underset{\sim}{u}^{*\prime} \cdot \underset{\sim}{\nabla}^* z^* = \nabla^{*2} T^{*\prime}$$

and

$$\frac{\partial \underset{\sim}{u}^{*\prime}}{\partial t^*} = -\underset{\sim}{\nabla}^* p^{*\prime} + \underset{\sim}{\hat{k}} \underbrace{\frac{\beta g d^3 \Delta T}{\alpha^2}} T^{*\prime} + \frac{\nu}{\alpha} \nabla^{*2} \underset{\sim}{u}^{*\prime}$$

$$\hookrightarrow = \left[\frac{\beta \Delta T g d^3}{\alpha \nu}\right]\left[\frac{\nu}{\alpha}\right]$$

$$= Ra\, Pr$$

After this pt, drop *'s

To solve this problem, we wish to reduce the # of dep variables on which problem depends

Take curl of mom. eq'n ⟹

$$\frac{\partial \underset{\sim}{\omega}'}{\partial t} = Ra\, Pr \left( \underset{\sim}{\nabla} \times \left( \hat{\underset{\sim}{k}}\, T' \right) \right) + Pr\, \nabla^2 \underset{\sim}{\omega}'$$

$$= Ra\, Pr \left( \underset{\sim}{\nabla} T' \times \hat{\underset{\sim}{k}} \right) + Pr\, \nabla^2 \underset{\sim}{\omega}'$$

Take curl of this equation:

$$\frac{\partial}{\partial t} \left( \underset{\sim}{\nabla} \times \underset{\sim}{\omega}' \right) = Ra\, Pr \left( \underset{\sim}{\nabla} \times \left( \underset{\sim}{\nabla} T' \times \underset{\sim}{k} \right) \right) + Pr\, \nabla^2 \underset{\sim}{\nabla} \times \underset{\sim}{\omega}'$$

Now $\underset{\sim}{\nabla} \times \underset{\sim}{\omega}' = \underset{\sim}{\nabla} \times \left( \underset{\sim}{\nabla} \times \underset{\sim}{u}' \right) \equiv \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm} \frac{\partial}{\partial x_l} u_m$

in Einstein notation

explain meaning of $\varepsilon_{ijk} \Rightarrow 0$ if $i=j$, $i=k$, $j=k$

1 if cyclic

$-1$ if counter cyclic

$$= \left( \varepsilon_{ijk} \varepsilon_{klm} \right) \frac{\partial^2 u_m}{\partial x_j \partial x_l} = \left( \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) \frac{\partial^2 u_m}{\partial x_j \partial x_l}$$

$$= \frac{\partial}{\partial x_i} \left( \frac{\partial u_j}{\partial x_j} \right) - \frac{\partial^2 u_i}{\partial x_j^2} \equiv \underset{\sim}{\nabla} \left( \cancel{\underset{\sim}{\nabla} \cdot \underset{\sim}{u}} \right) - \nabla^2 \underset{\sim}{u}$$

0 from C.E

and

$$\underset{\sim}{\nabla} \times \left( \underset{\sim}{\nabla} T' \times \hat{\underset{\sim}{k}} \right) = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm} \frac{\partial T'}{\partial x_l} \delta_{m3}$$

$$= \varepsilon_{ijk} \varepsilon_{klm} \delta_{m3} \frac{\partial^2 T'}{\partial x_j \partial x_l} = \frac{\partial}{\partial x_i} \frac{\partial T'}{\partial x_3} - \delta_{i3} \frac{\partial^2 T'}{\partial x_j^2}$$

Thus;

$$\frac{\partial}{\partial t}\nabla^2 \underset{\sim}{u}' = Ra\,Pr\left(\hat{\underset{\sim}{k}}\,\nabla^2 T' - \underset{\sim}{\nabla}\frac{\partial T'}{\partial z}\right) + Pr\,\nabla^4 \underset{\sim}{u}$$

and in particular $[\quad]\cdot\hat{\underset{\sim}{k}} \equiv$

End Lec. 17

$$\frac{\partial}{\partial t}\nabla^2 W' = Ra\,Pr\underbrace{\left(\frac{\partial^2 T'}{\partial x^2}+\frac{\partial^2 T'}{\partial y^2}\right)}_{\nabla_1^2 T'}+Pr\,\nabla^4 W'$$

Recall energy eq'n:

$$\frac{\partial T'}{\partial t} - \underset{\sim}{u}'\cdot\underset{\sim}{\nabla} z = \nabla^2 T'$$

or $W' = \frac{\partial T'}{\partial t} - \nabla^2 T'$ ( $\nabla_1^2$

∴ have 2 eq'ns w/ 2 dep. variables. Now manipulate to eliminate $T'$;

Take $\frac{\partial}{\partial t}$ & $\nabla^2$ of mom. eq'n & subtract to obtain:

$$\frac{\partial^2}{\partial t^2}\nabla^2 W' - \frac{\partial}{\partial t}\nabla^4 W' = Ra\,Pr\,\nabla_1^2\left[\frac{\partial T'}{\partial t} - \nabla^2 T'\right] + Pr\left(\nabla^4\frac{\partial W'}{\partial t} - \nabla^6 W\right)$$

$$= Ra\,Pr\,\nabla_1^2 W' + Pr\left(\nabla^4\frac{\partial W'}{\partial t} - \nabla^6 W'\right)$$

or:

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 W' = Ra\,\nabla_1^2 W'$$

by rearrangement

If we had eliminated $w'$, would have obtained identical eq'n for $T'$

Now we need some B.C.'s

At $z=0,1$, have zero normal velocity

$\therefore$ $w'=0$ at $z=0,1$

May have rigid or free bdy's $\Rightarrow$ focus on free bdy's since sol'n is simpler

For a free surface, tangential surface stress is zero

$$\therefore \left( \frac{\partial u}{\partial z} + \underbrace{\cancel{\frac{\partial w}{\partial x}}}_{0 \text{ since } w=0} \right)\Bigg|_{\text{bdy}} = 0 \quad , \quad \text{sim. obtain } \frac{\partial v}{\partial z} = 0$$

Thus, from C.E.: $\underset{\sim}{\nabla} \cdot \underset{\sim}{u}' = 0 \equiv \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$

or, taking $z$ deriv:

$$\frac{\partial^2 w'}{\partial z^2} = -\left( \frac{\partial}{\partial x}\frac{\partial u'}{\partial z} + \frac{\partial}{\partial y}\frac{\partial v'}{\partial z} \right) = 0 \quad \text{at } z=0,1$$

we also take $T'=0$ at $0,1$ as well $\Rightarrow$ use to obtain last two B.C.'s — have 6, necessary # for 6th order eq'n

Also need I.C.'s.

What do we use? ⇒ worry about later

Problem & B.C.'s are symmetric in x&y

Assume a normal mode of disturbance of form: (seek sol'ns of this form)

$$w' = W(z) f(x,y) e^{st}, \quad T' = G(z) f(x,y) e^{st}$$

where $s = \sigma + i\omega$ ⇒ complex

for a small disturbance, will either grow $(\sigma > 0)$, or decay $(\sigma < 0)$, or be neutrally stable $(\sigma = 0)$

Plug into eq'n of energy

$$\frac{\partial T'}{\partial t} - w' = \nabla^2 T'$$

$$\therefore sGfe^{st} - Wfe^{st} = G\nabla_1^2 f e^{st} + G'' f e^{st}$$

$$\text{or: } -G''f + sGf - Wf = G\nabla_1^2 f$$

dividing by Gf:

$$-\frac{G''}{G} + s - \frac{W}{G} = \frac{1}{f}\nabla_1^2 f = \underline{-a^2}$$

choose - to seek periodic sol'ns

Thus: $$\left(\frac{\partial^2}{\partial z^2} - a^2 - s\right) G = -W$$

and $\nabla_1^2 f + \underset{\uparrow \text{arb.}}{a^2} f = 0$ } reduced wave eq'n

$a \equiv$ horizontal wave # of disturbance

We had the mom. eq$^{n}$:

$$\frac{\partial}{\partial t}\nabla^2 w' - P_r \nabla^4 w' = Ra\, P_r \nabla_1^2 T'$$

plugging in for normal forms of $w'$, $T'$, obtain:

$$s\left(W\nabla_1^2 f + f D^2 W\right)e^{st} - P_r\left(W\nabla_1^4 f + 2D^2 W\nabla_1^2 f + f D^4 W\right)e^{st} = Ra\, P_r\, G\, \nabla_1^2 f e^{st}$$

where $D^2 \equiv \frac{\partial^2}{\partial z^2}$

Since $\nabla_1^2 f = -a^2 f$ we obtain:

$$s\left(D^2 - a^2\right)W - P_r\left(D^4 - 2D^2 a^2 + a^4\right)W = -a^2 Ra P_r G$$

Now at $z = 0, 1$ $W, D^2 W, G = 0$ (B.C.$^{s}$)

∴ by elimination $D^4 W = 0$ as well.

we may do the same to our final eq$^{n}$:

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{1}{P_r}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w' = Ra\, \nabla_1^2 w'$$

$$\therefore \left(s - D^2 + a^2\right)\left(\frac{s}{P_r} - D^2 + a^2\right)\left(D^2 - a^2\right)W = -a^2 Ra\, W$$

w/ B.C.$^{s}$ $W = D^2 W = D^4 W = 0$ at $z = 0, 1$

is sixth order eigen value problem for countable infinity of eigenvalues $s_n$ & eigenf$^{\underline{n}}$s $W_n(z)$

$$S_n = f^{\underline{n}}(a, Ra, P_r)$$

Problem will be <u>unstable</u> if $\sigma \equiv \mathrm{Re}(s) > 0$ for <u>any</u> $n$ or <u>any</u> $a$

& stable if $\sigma \leq 0$ for <u>all</u> $n, a$

critical $Ra = Ra_c$ is value for which

$\sigma(a, Ra, P_r) > 0$ for some $a$ if $Ra > Ra_c$

<u>and</u> $\sigma(a, Ra, P_r) \leq 0$ for <u>all</u> $a$ if $Ra \leq Ra_c$

Now solve for eigen f<u>n</u>s & eigenvalues:

problem has very simple sol'n:

often $W_n$ very difficult to obtain — variational principle allows us to estimate $a_{cr}$, $Ra_{cr}$ if guess form of $W_n$ close to actual eigenf<u>n</u>

$W_n = \sin n\pi z$, $n = 1, 2, 3 \ldots$

Thus we have the eigenvalue relation:

$$\left(s_n + n^2\pi^2 + a^2\right)\left(\frac{s_n}{P_r} + n^2\pi^2 + a^2\right)\left(n^2\pi^2 + a^2\right) = a^2 Ra$$

Which is a quadratic eq'n for $s_n$:

$$s_n = -\frac{1}{2}(1 + P_r)\left(n^2\pi^2 + a^2\right) \pm \left\{\frac{1}{4}\left(P_r - 1\right)^2\left(n^2\pi^2 + a^2\right)^2 + \frac{a^2 Ra P_r}{\left(n^2\pi^2 + a^2\right)}\right\}^{1/2}$$

Now if $Ra \leq 0$, $\mathrm{Re}(s_n) < 0$ for all $n, a, P_r$

which makes sense physically: favorable $\Delta T$

For marginal stability, $Re(s) = 0$

For this problem, we can prove exchange of stabilities: if $\sigma = 0$ then $\omega = 0$

$\therefore$ when $Re(s) = 0$, $s = 0$

Hence $s = 0$ when

$$(n^2\pi^2 + a^2)^3 = a^2 Ra$$

or $$Ra = \frac{(n^2\pi^2 + a^2)^3}{a^2}$$

we wish to det. smallest value of Ra for which problem is neutrally stable

1st, note above is inc. fn (n) $\therefore$ most unstable mode is lowest $n$ (i.e. $n=1$)

$$\therefore\ Ra_{cr} = \min \frac{(\pi^2 + a^2)^3}{a^2}$$

find $a_{cr}$ by taking $$\frac{\partial Ra}{\partial a^2} = 0 \equiv -\frac{(\pi^2 + a^2)^3}{a^4} + 3\frac{(\pi^2 + a^2)}{a^2}$$

or $$\frac{\pi^2 + a_{cr}^2}{a_{cr}^2} = 3 \quad \therefore\ a_{cr} = \frac{\pi}{\sqrt{2}}$$

w/ $$Ra_{cr} = \frac{\left(\frac{3\pi^2}{2}\right)^3}{\frac{1}{2}\pi^2} = \frac{27\pi^4}{4} = 657.5$$

w/ horizontal wavelength at critical conditions of: $$\lambda = \frac{2\pi d}{a_{cr}} = 2\sqrt{2}\, d = 2.83\, d$$

Note: critical Ra indep. of Pr, but not growth & damping

Problem is more d.f. for solid bdy's ⇒ more complex B.C.'s on $W(z)$ & cannot analytically det. $S_n$ ⇒ find numerically (transcendental eq'n)

What do cells look like? arise from sol'n of $\nabla_1^2 f = -a^2 f$

Can obtain profile for rolls, square cells, rectangles, hexagons

Result of non-linear int, also walls ⇒ fastest growing disturbance will win out if non-linear effects don't dominate.

For rigid-free, $Ra_{cr} = 1101$

& rigid-rigid, $Ra_{cr} = 1708$

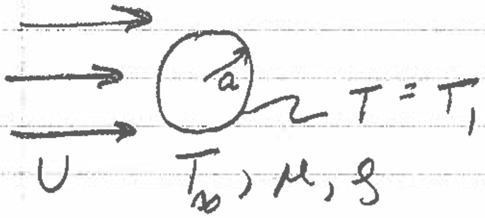
For HW, will solve analogous problem for cst ht generation.

End Lec. 18

# Forced Convection Past a Heated Sphere

⇒ classic problem, illustrates technique of matched asymptotic expansion

Ref: Acrivos & Taylor, Phys. Fluids 5, 387 (1962)



know that $T^* \equiv \dfrac{T - T_\infty}{T_1 - T_\infty} = f^n(Re, Pr)$

from dimensional analysis.

Look at case where $Re = \underline{\underline{0}}$ (creeping flow)

we thus <u>know</u> sol'n to eq'n of motion ⇒ just stoke's flow past sphere. Left w/:

$$\nabla^{*2} T^* = Pe\, \underset{\sim}{u}^* \cdot \underset{\sim}{\nabla}^* T^*$$

where $Pe \equiv \dfrac{\text{conv.}}{\text{cond.}} = \dfrac{Ua}{\alpha} = Re\, Pr$

can have $Re \ll 1$ & $Pe = O(1)$ or even $\gg 1$ as $Pr$ <u>very</u> large for viscous fluids, but look at limit:

$$Re \ll Re\, Pr \ll 1$$

Let $Pe \equiv \varepsilon \ll 1$ (slow flow) have B.C.'s: $T^* = 1$ at $r^* = 1$,

$T^* = 0$ as $r^* \to \infty$

problem has axial symmetry ⇒ use spherical polar coordinates

$r$, $\mu \equiv \cos\theta$ ~ meas. from rear stagnat point

Eq'n becomes:

$$\nabla_r^2 T \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial T}{\partial \mu}\right]$$

$$= \varepsilon\left\{\left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right)\mu\frac{\partial T}{\partial r} + \frac{1-\mu^2}{r}\left(1 - \frac{3}{4r} - \frac{1}{4r^3}\right)\frac{\partial T}{\partial \mu}\right\}$$

where we have substituted the velocity profile for creeping flow past a sphere.

Now when $\varepsilon = 0$, assume that convection is negligible & obtain sol'n (harmonic)

$$\nabla^2 T = 0 \quad \therefore \quad T_0 = \frac{1}{r} \quad \text{(satisfies B.C.'s)}$$

What is next order term?

Let $T = \frac{1}{r} + \varepsilon T_1 + \cdots$

Then:

$$\nabla_r^2 T_1 = -\frac{\mu}{r^2}\left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right) \Big\} O(\varepsilon) \text{ problem}$$

which has the particular sol'n:

$$T_1^p = \left(\frac{1}{2} - \frac{3}{4r} - \frac{1}{8r^3}\right)\mu$$

but 1st term is just $\frac{1}{2}\mu$ at $\infty$

Can't kill off w/ harmonics ⇒ term which is cst at $\infty$ not prop. to $\mu$

Won't work!

Problem: leading order sol'n was not uniformly valid approx. at $\infty$ ⇒ just like flow past a sphere

Far from sphere, temp distrib. governed by:

$$\varepsilon \underset{\sim}{U} \cdot \underbrace{\underset{\sim}{\nabla} T}_{\sim \frac{1}{r}} = \underbrace{\nabla^2 T}_{\sim \frac{1}{r^2}}$$

terms balance for $r \sim O(1/\varepsilon)$

How do we solve problem? Divide into two regions:

$$0 < r \leq O\left(\tfrac{1}{\varepsilon}\right), \quad r \geq O\left(\tfrac{1}{\varepsilon}\right)$$

Then match sol'ns for $r = O(\frac{1}{\varepsilon})$ => region of overlap: method of matched asymptotic expansions.

Inner Region:

let $T(r, \mu, \varepsilon) = \frac{1}{r} + \sum_{n=1}^{\infty} f_n(\varepsilon) T_n(r, \mu)$

where $f_1 \to 0$ as $\varepsilon \to 0$ & $f_{n+1}/f_n \to 0$ as $\varepsilon \to$

We suppose (justify later) $f_1 = \varepsilon$

$\therefore$

$$T_1 = \left(\frac{1}{2} - \frac{3}{4r} - \frac{1}{8r^3}\right)\mu + \sum_{n=0}^{\infty} \left\{A_n r^n + B_n r^{-(n+1)}\right\} P_n($$

↳ particular sol'n ↳ homogeneous sol'n

where $P_n(\mu) \equiv$ Legendre polynomials

$P_0 = 1, \quad P_1 = \mu, \quad P_2 = \frac{1}{2}(3\mu^2 - 1), \quad \ldots$

We have B.C. at $r = 1$ in inner region:

$T_1 = \underline{0}$ at $r = 1$ (i.e. $T|_{r=1} = 1 + f_1 T_1 + \cdots = 1$

Thus $A_n = -B_n, \quad n = 0, 2, 3, \ldots$

& $A_1 + B_1 = \frac{3}{8}$

We may determine $B_n$ from matching condition w/ outer region.

Outer sol'n:

Choose new length scale.

Let $\rho = \varepsilon r$, $\therefore$ $\rho = O(1)$ in region of overlap

Hence

$T^{o}(\rho, \mu, \varepsilon)$:

$$\nabla_\rho^2 T^o \equiv \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial T^o}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial T^o}{\partial \mu}\right]$$

$$= \left(1 - \frac{3\varepsilon}{2\rho} + \frac{\varepsilon^3}{2\rho^3}\right)\mu\frac{\partial T^o}{\partial \rho} + \frac{1-\mu^2}{\rho}\left(1 - \frac{3\varepsilon}{4\rho} - \frac{\varepsilon^3}{4\rho^3}\right)\frac{\partial}{\partial}$$

$\uparrow$ Velocity profile in terms of $\rho$ $\uparrow$

Let $T^o = \sum_{n=0}^{\infty} F_n T_n^o(\rho, \mu)$

$\therefore$ to leading order simply set $\varepsilon = 0$ in eq'n

Thus: $\nabla_\rho^2 T_0^o = \mu \dfrac{\partial T_0^o}{\partial \rho} + \dfrac{1-\mu^2}{\rho}\dfrac{\partial T_0^o}{\partial \mu}$

(just eq'n w/ free stream velocity)

Sol'n to eq'n which satisfies B.C. at $\infty$ is:

$$F_0(\varepsilon)T_0^{\,o}(\rho,\mu) = \frac{F_0(\varepsilon)}{\rho} e^{-\frac{\rho}{2}(1-\mu)} \sum_{n=0}^{\infty} C_n P_n(\mu) \sum_{m=0}^{n} \frac{(n+m)!}{(n-m)!\,m!} \rho^{-1}$$

we require this to match w/ inner sol'n:

$$T = \frac{1}{r} + \varepsilon\left(\frac{1}{2} - \frac{3}{4r} - \frac{1}{8r^3}\right)\mu + \sum_{n=0}^{\infty}\left\{A_n r^n + B_n r^{-(n+1)}\right\}P_n$$

as $r \to O\left(\frac{1}{\varepsilon}\right)$ and $\rho \to O(\varepsilon)$

Rewriting outer sol'n in inner variables:

$$F_0(\varepsilon)T_0^{\,o}(\varepsilon r,\mu) = \frac{F_0(\varepsilon)}{\varepsilon r} e^{-\frac{\varepsilon r}{2}(1-\mu)} \sum_{n=0}^{\infty} C_n P_n(\mu) \sum_{m=0}^{\infty} \frac{(n+m)!}{(n-m)!\,m!} \left(\frac{1}{\varepsilon r}\right)$$

This will only match leading term of inner sol'n $\left(\frac{1}{r}\right)$ if $C_n = 0$ for $n \neq 0$, $C_0 = 1$ and $F_0(\varepsilon) = \varepsilon$

Thus outer sol'n becomes:

$$T^{\,o}(\varepsilon r,\mu) \approx \frac{1}{r} e^{-\frac{\varepsilon r}{2}(1-\mu)} + O(F_1(\varepsilon))$$

$$\approx \frac{1}{r}\left[1 - \frac{\varepsilon r}{2}(1-\mu) + \cdots\right] + O(F_1(\varepsilon))$$

which must match inner sol'n:

$$T \sim \frac{1}{r} + \frac{1}{2}\varepsilon\mu + \sum_{n=0}^{\infty} A_n r^n P_n(\mu) \text{ for large } r$$

Thus we see that $\frac{\varepsilon}{2}\mu$ term matches automatically. Also, matching only works if $A_n = 0 \; n \neq 0$ and $A_0 = -\frac{1}{2}$

Thus in the inner region:

$$T = \frac{1}{r} + \varepsilon\left\{\frac{1}{2}\left(\frac{1}{r} - 1\right) + \mu\left(\frac{1}{2} - \frac{3}{4r} + \frac{3}{8r^2} - \frac{1}{8r^3}\right)\right\}$$
$$+ O(f_2)$$

w/ $Nu = \dfrac{Q}{4\pi a\, k(T_0 - T_\infty)} = \dfrac{1}{2}\displaystyle\int_{-1}^{1} -\left(\frac{\partial T}{\partial r}\right)_{r=1} d\mu$

w/ $Q$ = total rate of ht. transf.,

obtain $Nu = 1 + \frac{\varepsilon}{2} + O(f_2)$

Interesting to note that $f_2 \neq \varepsilon^2$, rather obtain term $\varepsilon^2 \log \varepsilon$ which decays away more slowly.

Result above can be generalized to bodies of arbitrary shape, e.g.:

$$Nu = Nu\Big|_{\varepsilon=0}\left(1 + \frac{\varepsilon}{2}\right)$$

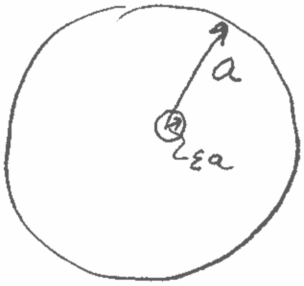
End lec. 19

Lec 20 = test, Lec. 21 = review test

Let's look at HW problem ⇒ will give new one for next ~~Monday~~ Friday

Flow thru a tube w/ an Axial Wire

Ref: Perturbation Methods, VanDyke



We have eq'n:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{\mu}\frac{\partial p}{\partial z} = cst$$

render Dimensionless. Let $r^* = r/a$,

$$u^* = \frac{+u}{\left(\frac{a^2}{\mu}\frac{\partial p}{\partial z}\right)}$$

$$\therefore \quad \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u^*}{\partial r^*}\right) = -1$$

w/ $u^* = 0$ at $r^* = 1$, $r^* = \varepsilon$ where $\varepsilon \ll 1$

we may solve this directly, but let's use method of matched asymptotic exp. instead:

wire thru eliptical tube not solvable analytically, but may be solved asymptotically.

So: Problem has 2 regions ⇒ Global region where $r^* = O(1)$, inner region where $r^* = O(\varepsilon)$ & problem must be rescaled.

Look at outer region:

sol'n for $\varepsilon = 0$ is just (dropping $*$'s)

$$u = \frac{1}{4}(1 - r^2) + o(1) \rightarrow \text{little } o \text{ means of smaller mag. than}$$

which is not valid as $r \rightarrow \varepsilon$

In inner region:

let $\rho = r/\varepsilon$

$\therefore$ have eq'n:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) = -\varepsilon^2$$

To leading order in inner region, just have

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) = 0$$

$$\therefore\ u = C_1 + \overset{f_0(\varepsilon)}{\cancel{C_2}} \ln \rho$$

w/ B.C. that $u\big|_{\rho=1} = 0 \ \therefore\ C_1 = 0$

and $u = C_2 \ln \rho$ (note: $C_2$ may be $f^{\underline{n}}(\varepsilon)$)

as was true in last proble
i.e. $u = f_0(\varepsilon)$
$+ f_1(\varepsilon$
$+$

We must match this sol'n w/ outer sol'n as

$\rho \rightarrow \frac{1}{\varepsilon}$, $r \rightarrow \varepsilon$

writing inner sol'n in outer variables:

$$u = C_2 \ln(r/\varepsilon) = C_2\left[\ln(r) + \ln(1/\varepsilon)\right]$$

$\therefore u = c_2 \ln\left(\frac{1}{\varepsilon}\right)\left[1 + \frac{\ln(r)}{\ln(1/\varepsilon)}\right]$

Now as $r \to O(\varepsilon)$ outer sol'n becomes

$u \approx \frac{1}{4}$ $\quad\left( u = \frac{1}{4}(1-(\varepsilon \rho)^2) \Rightarrow \frac{1}{4} + O(\varepsilon^2)\right)$ in inner region

$\therefore$ to match, $c_2 = \dfrac{1}{4\ln(1/\varepsilon)}$

Thus inner sol'n is of form:

$$u = \frac{1}{4}\,\frac{\ln(r/\varepsilon)}{\ln(1/\varepsilon)} = \frac{1}{4}\left(1 + \frac{\ln(r)}{\ln(1/\varepsilon)}\right)$$

We may construct a uniformly valid approx to sol'n from inner & outer sol'n:

= inner sol'n + outer sol'n − lim of outer sol'n in inner region (to correct order)

$$\therefore u \approx \frac{1}{4}\left(1 + \frac{\ln(r)}{\ln(1/\varepsilon)}\right) + \frac{1}{4}(1-r^2) - \frac{1}{4}$$

$$= \frac{1}{4}(1-r^2) + \frac{1}{4}\,\frac{\ln(r)}{\ln(1/\varepsilon)}$$

which is very similar to exact sol'n:

$$u = \frac{1}{4}(1-r^2) + \frac{1}{4}\,\frac{\ln(r)}{\ln(1/\varepsilon)}(1-\varepsilon^2)$$

Now look at another problem: Unsteady cond. from a cylinder.

We wish to det. T profile & Q' from inf. cylinder at temp $T_1$ in solid w/ $\alpha$ & temp $T_0$ at long times.

Look at steady problem first:

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0$$

let $t^* = \frac{t\alpha}{a^2}$, $r^* = r/a$, $T^* = \frac{T-T_0}{T_1-T_0}$ $\therefore T^* \to 0$ as $r^* \to \infty$

sol'n to steady problem:

$$\left.T^* = C_1 + C_2 \ln r^* \right\} \text{at } r^*=1,\ T^*=1$$

$\therefore T^* = 1 + C_2 \ln r^*$

Can't match condition at $\infty$.

Now for unsteady problem:

Convenient to introduce new time variable s:

$$s = \frac{1}{2t^{*1/2}}$$

$\therefore$ interested in problem when $s \to 0$

Now have eq'n: (drop all *'s):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + 2s^3\frac{\partial T}{\partial s} = 0$$

since $\frac{\partial}{\partial t} = \frac{\partial}{\partial s}\frac{\partial s}{\partial t} = -2s^3\frac{\partial}{\partial s}$

For small $s$ we have inner problem:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=0$$

w/ inner solution: $T = 1 + C_2 \ln r$

where $C_2$ will be $f^{\underline{n}}(s)$ to be determined.

In outer region, terms

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right) \quad \& \quad 2s^3\frac{\partial T}{\partial s} \quad \text{balance}$$

$$\sim \frac{T}{r^2} \qquad\qquad \sim s^2 T$$

∴ in outer region, rescale $r$ w/ $s$

and we let $\rho = rs$ ← render dim by $\rho = \frac{r}{(4\alpha t)^{1/2}}$

w/ eq'n in terms of $\rho, s$ : <u>not</u> orthogonal coordinates, which complicates matters.

$$\left(\frac{\partial T}{\partial r}\right)_s = s\left(\frac{\partial T}{\partial \rho}\right)_s ,$$

similarly

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right)_s = s^2\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial T}{\partial \rho}\right)\right)_s$$

and we also have:

$$\left(\frac{\partial T}{\partial s}\right)_r = \left(\frac{\partial T}{\partial s}\right)_\rho + \left(\frac{\partial T}{\partial \rho}\right)_s \frac{\partial \rho}{\partial s}$$

$$= \left(\frac{\partial T}{\partial s}\right)_\rho + \left(\frac{\partial T}{\partial \rho}\right)_s \frac{\rho}{s}$$

Thus:

$$s^2\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial T}{\partial \rho}\right)\right) + 2\rho s^2 \frac{\partial T}{\partial \rho} + 2s^3\frac{\partial T}{\partial s} = 0$$

or $$T_{\rho\rho} + \left(\frac{1}{\rho} + 2\rho\right) T_\rho + 2s\frac{\partial T}{\partial s} = 0$$

which, to leading order as $s \to 0$, gives:

$$T_{\rho\rho} + \left(\frac{1}{\rho} + 2\rho\right) T_\rho = O(s)$$

with outer sol'n B.C. $T \to 0$ as $\rho \to \infty$

Integrating:

$$T_\rho = c_1 e^{-(\ln \rho + \rho^2)} = \frac{c_1}{\rho} e^{-\rho^2}$$

$$\therefore \; T = \int_\rho^\infty \frac{-c_1}{x} e^{-x^2} dx$$

let $z = x^2$, $\therefore \frac{dx}{x} = \frac{1}{2}\frac{dz}{z}$

and $$T^0 = \int_{\rho^2}^{\infty} \frac{C}{z} e^{-z} dz \equiv C E_1(\rho^2)$$

we now need to match inner & outer solutions:

$$\lim_{z \to 0} E_1(z) = -\log(z) + O(1)$$

$$\therefore T^0(r, s) \Rightarrow -2C \log(r s) + C \cdot O(1)$$

$$= -2C \log s \left( 1 + \frac{\log r}{\log s} + O\left(\frac{1}{\log s}\right) \right)$$

which must match inner sol'n:

$$T = 1 + c_2 \log r$$

$$\therefore C = \frac{-1}{2\log(s)} = \frac{1}{2\log(1/s)}$$

and $$c_2 = \frac{-1}{\log(1/s)}$$

$\therefore$ we have the inner solution

$$T = 1 - \frac{\log(r)}{\log(1/s)}$$

and the outer solution:

$$T = \frac{1}{2\log(1/\varepsilon)} E_1((\varepsilon r)^2)$$

The energy loss from the cylinder is thus:

$$\overline{Nu} = -\frac{\partial T}{\partial r}\Big|_{r=1} = \frac{1}{\log(1/s)} = \frac{1}{\frac{1}{2}\log\left(\frac{4\alpha t}{b^2}\right)} \quad \text{as } t \to \infty$$

End lec. 22

---

Now turn to conduction in solids:

Consider sphere, temp $T=1$, fluid $T=0$

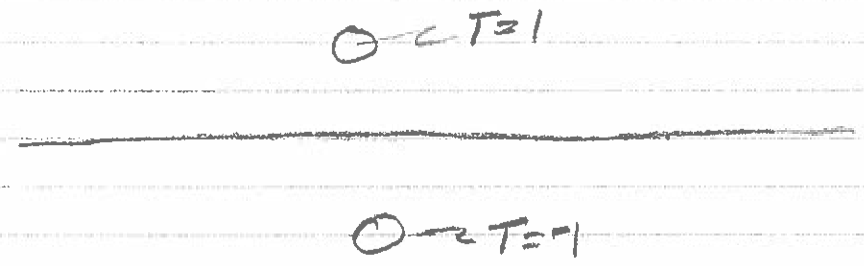
wish to find change in energy loss due to plane:

How do we obtain temp. distrib?

Could solve directly: use bispherical coord ⇒ very messy.

Instead, suppose $a/h \ll 1$, find asymptotic representation: use method of reflections: Useful in wide variety of linear problems: problem is just sum of simple solutions.

let sphere be at $y_i$ : problem is identical to that if second sphere is at <u>image point</u> — $-y_i$. If plane were curved, image point would have complex location. Thus:



normalize all distances w.r.t. sphere radius

$\therefore\ \nabla^2 T = 0,\ T = 1, |x_i - y_i| = 1\ ;\ T = -1, |x_i + y_i| = 1$

will take T as sum of $T^{(1)} \Rightarrow$ temp distrib. produced by sphere at $y_i$ & $T^{(2)}$ from sphere at $-y_i$

To leading order, $T^{(1)}$ & $T^{(2)}$ just that of isolated spheres, thus:

$$T^{(1,0)} = \frac{1}{|x_i - y_i|}\ ,\quad T^{(2,0)} = \frac{-1}{|x_i + y_i|}$$

and $T^{(0)} = T^{(1,0)} + T^{(2,0)}$

Note that as $x_i >> y_i$, $T^{(0)} \to O(\frac{a^2}{r^2})$ rather than $O(\frac{a}{r})$ for an isolated sphere

Now $T^{(0)}$ satisfies B.C.'s on neither sphere

$\therefore$ gives rise to a series of reflections:

$T^{(1,1)}$ s.t $\nabla^2 T^{(1,1)} = 0$; $T^{(1,1)} = -T^{(2,0)}$ on $|x_i - y_i| =$

and $T^{(2,1)}$ s.t. $\nabla^2 T^{(2,1)} = 0$; $T^{(2,1)} = -T^{(1,0)}$ on $|x_i + y_i| =$.

Now $T^{(2,0)}$ in vicinity of sphere 1 is rather complicated, so it is customary to construct an asymptotic solution: Expand in a Taylor series

$$T^{(2,0)} = T^{(2,0)}\Big|_{y_i} + \frac{\partial T^{(2,0)}}{\partial x_j}\Big|_{y_i} \cdot (x_j - y_j)$$

$$+ \frac{1}{2} \frac{\partial^2 T^{(2,0)}}{\partial x_j \partial x_K} \cdot (x_j - y_j)(x_K - y_K) + \ldots$$

Now $T^{(2,0)} = \dfrac{-1}{|x_i + y_i|}$

$\therefore$ first term is $O(\frac{a}{h})$, second $= O\left(\frac{a^2}{h^2} |x_i - y_i|\right)$, et

just look at first term:

We expand $T^{(1,1)} = T^{(1,1,0)} + T^{(1,1,1)} + \ldots$

where each succeeding term is of higher order in $\frac{a}{h}$:

$\nabla^2 T^{(1,1,0)} = 0$; $T^{(1,1,0)} = -T^{(2,0)}\Big|_{y_i}$ on $|x_i - y_i| = 1$

$\nabla^2 T^{(1,1,1)} = 0$; $T^{(1,1,1)} = -\frac{\partial T^{(2,0)}}{\partial x_j}\Big|_{y_i} (x_j - y_j)$ on , et

Will obtain similar series of problems for sphere (2)

Let's just solve to leading order:

$$T^{(1,1,0)} = \frac{1}{|2y_i|}\,\frac{1}{|x_i - y_i|} \quad \swarrow O\left(\frac{a}{h}\right)$$

and

$$T^{(2,1,0)} = \frac{-1}{|2y_i|}\,\frac{1}{|x_i + y_i|} \quad \swarrow O\left(\frac{a^2}{h^2}\right) \text{ at sphere(1)}$$

so to first order can neglect!

Thus near sphere (1):

$$T = \frac{1}{|x_i - y_i|} - \frac{1}{|x_i + y_i|} + \frac{1}{|2y_i|}\,\frac{1}{|x_i - y_i|} + O\left(\frac{a^2}{h^2}\right)$$

with the second term having the expansion:

$$\frac{-1}{|x_i + y_i|} = \underbrace{\frac{-1}{|2y_i|}}_{\sim O\left(\frac{a}{h}\right)} + \underbrace{\frac{1}{|x_i + y_i|^3}(x_j + y_j)(x_j - y_j)}_{\sim O\left(\frac{a^2}{h^2}\right)} + \ldots$$

$$\therefore\ T = \frac{1}{r} - \frac{1}{2}\frac{a}{h} + \frac{1}{2}\frac{a}{h}\frac{1}{r}$$

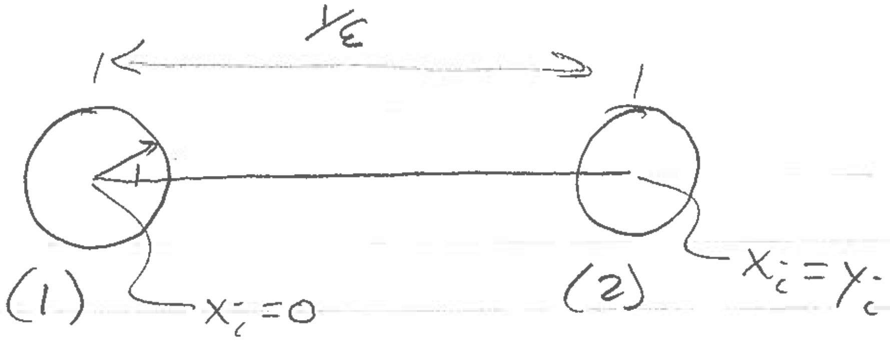
$$= \frac{1}{r} - \frac{1}{2}\frac{a}{h}\left(1 - \frac{1}{r}\right)$$ w/ $r$ centered on sphere(1

Thus energy loss is increased by a factor

$$Nu = 1 + \frac{1}{2}\frac{a}{h} + O\left(\frac{a^2}{h^2}\right)$$

higher order terms get rapidly more complex

# Method of Reflections: Ghostsphere



$$\nabla^2 T = 0, \quad T\Big|_{|x_i|=1} = 1 \qquad T\Big|_{|x_i|\to\infty} = 0$$

$$T\Big|_{|x_i - y_i| = 1} = C \quad (cst) \; ; \qquad \int_{|x_i - y_i| = 1} \underset{\sim}{q} \cdot \underset{\sim}{n} \, dA = 0$$

let $r = |x_i|$, $r_2 = |x_i - y_i|$

∴ condition on second sphere becomes:

$$T\Big|_{r_2 = 1} = C \quad , \quad \int_{r_2 = 1} \frac{\partial T}{\partial r_2} \, dA = 0$$

We have the solution to the first sphere in absence of the second:

$$T^{(1,0)} = \frac{1}{r}$$

which satisfies the condition on (2)

$$\int_{r_2=1} \underset{\sim}{q}^{(1,0)} \cdot \underset{\sim}{n} \, dA = \int_{r_2=1} \underset{\sim}{\nabla} T^{(1,0)} \cdot \underset{\sim}{n} \, dA = \int_{r_2<1} \nabla^2 T^{(1,0)} \, dV = 0$$

but <u>not</u> the condition $T = C$ on $r_2 = 1$

This gives rise to the image temperature distributi

$$\nabla^2 T^{(2,1)} = 0 \; ; \quad T^{(2,1)}\Big|_{r_2=1} = C - T^{(1,0)}\Big|_{r_2=1}$$

where C must be determined from the condition

$$\int_{r_2=1} \frac{\partial T^{(2,1)}}{\partial r_2} \, dA = 0$$

The temperature distribution is invariant w.r.t. rotation about the axis joining the line of centers, thus we may expand $T^{(2,1)}$ in terms of the spherical harmonics (decaying):

$$T^{(2,1)} = \sum_{n=0}^{\infty} B_n \, r_2^{-(n+1)} \, P_n(\mu)$$

We also require the no flux condition:

$$\int_{r_2=1} \frac{\partial T^{(2,1)}}{\partial r_2}\, dA = 0 = \int_{r=1} \sum_{n=0}^{\infty} \frac{-(n+1)B_n}{r_2^{\,n+2}} P_n(\mu)\, dA$$

$$= 2\pi \sum_{n=0}^{\infty} -(n+1) B_n \int_{-1}^{1} P_n(\mu)\, d\mu \xrightarrow{\quad} \begin{array}{l} = 0 \text{ for } n \neq 0 \\ = 2 \text{ for } n = 0 \end{array}$$

$$= -4\pi B_0 \qquad \therefore \quad B_0 = 0 \quad !$$

hence $T^{(2,1)} = \sum_{n=1}^{\infty} B_n\, r_2^{-(n+1)} P_n(\mu)$

we may determine the $B_n$ by the B.C.

$$T^{(2,1)}\Big|_{r_2=1} = C - T^{(1,0)}\Big|_{r_2=1}$$

To obtain leading order solution, expand $T^{(1,0)}$ about $x_i = y_i$:

$$T^{(1,0)} = \underbrace{T^{(1,0)}\Big|_{x_i=y_i}}_{O(\varepsilon)} + \underbrace{\frac{\partial T^{(1,0)}}{\partial x_j}\Big|_{x_i=y_i} (x_j - y_j)}_{O(\varepsilon^2)} + \underbrace{\frac{1}{2}\frac{\partial^2 T^{(1,0)}}{\partial x_j \partial x_k}\Big|_{x_i=y_i} (x_j - y_j)(x_k -}_{O(\varepsilon^3)}$$

$+ \ldots$

We will need only first two terms

Now if we define $y_i = \frac{1}{\varepsilon}\delta_{i1}$

and $\left(x_1 - \frac{1}{\varepsilon}\right) = r_2 \mu$ ; $r_2 = |x_i - y_i|$

(i.e. $\mu = \cos\theta_2 = 1$ corresponds to $\delta_{i1}$ axis)

then :

$$T^{(1,0)}\Big|_{x_i=y_i} = \frac{1}{r}\Big|_{x_i=y_i} = \varepsilon$$

$$\frac{\partial T^{(1,0)}}{\partial x_j}\Big|_{x_i=y_i}(x_j-y_j) = -\frac{x_j}{r^3}\Big|_{x_i=y_i}(x_j-y_j) = -\varepsilon^2\delta_{j1}(x_j-y_j)$$

$$= -\varepsilon^2 r_2 \mu = -\varepsilon^2 r_2 P_1(\mu)$$

$$\frac{\partial^2 T^{(1,0)}}{\partial x_j \partial x_k}\Big|_{x_i=y_i}(x_j-y_j)(x_k-y_k) = \left[\frac{-\delta_{jk}}{r^3} + 3\frac{x_j x_k}{r^5}\right]\Big|_{x_i=\frac{1}{\varepsilon}\delta_{i1}}(x_j-y_j)(x_k-y_k)$$

$$= \varepsilon^3\left[-\delta_{jk} + 3\delta_{j1}\delta_{i1}\right](x_j-y_j)(x_k-y_k)$$

$$= \varepsilon^3 r_2^2(\mu^2-1) = 2\varepsilon^3 r_2^2 P_2(\mu)$$

Matching, we find:

$$C = \varepsilon$$

$$B_1 = -\varepsilon^2$$

$$B_2 = \varepsilon^3, \text{ etc.}$$

$$\therefore \; T^{(2,1,1)} = +\varepsilon^2 \frac{\mu}{r_2^{\,2}} = +\varepsilon^2 \frac{(x_1 - \frac{1}{\varepsilon})}{r_2^{\,3}}$$

which gives rise to the temp. at the first sphere:

$$T^{(2,1,1)}\Big|_{|x_i|=1} = -\varepsilon^4 + O(\varepsilon^5)$$

Hence we must have the first reflection off the first sphere:

$$T^{(1,1,0)} = +\frac{\varepsilon^4}{r}$$

and thus the Nusselt ~~#~~ :

$$Nu = 1 + \varepsilon^4 + O(\varepsilon^5)$$

11.0 a.

# Variational Methods

$K\nabla^2 T_S + S = 0 \Rightarrow$ steady ht cond w/ source

$T_S$ specified on $\partial D$

what is solution? May be complex

Obtain approximate sol'n via variational method

Consider integral:

$$I = \int_D \left[ (\underset{\sim}{\nabla} T)\cdot(\underset{\sim}{\nabla} T) - \frac{2}{K} S T \right] dV$$

Assert that $I$ is a minimum for $T = T_S$ out of all $T(\underset{\sim}{x})$ which satisfies B.C. on $\partial D$

Let's prove this:

$$\text{Let } T(\underset{\sim}{x}) = \underbrace{T_S(\underset{\sim}{x})}_{\hookrightarrow \text{right answer}} + \underbrace{T'(x)}_{\hookrightarrow \text{error}}$$

and show that $I$ increases for $T' \neq 0$

and we have $T'(\underset{\sim}{x}) = 0 \quad \underset{\sim}{x} \in \partial D$

So:

$$I = \int_D \left[\underset{\sim}{\nabla} T_s \cdot \nabla T_s - \frac{2}{K} S T_s\right] dV$$

$$+ \int_D \left[2\, \underset{\sim}{\nabla} T_s \cdot \underset{\sim}{\nabla} T' - 2 \frac{S}{K} T'\right] dV$$

$$+ \int_D \left(\underset{\sim}{\nabla} T' \cdot \underset{\sim}{\nabla} T'\right) dV$$

1st term is some # $I_s$

third term is positive definite

Now for 2nd term:

$$2 \int_D \left[\underset{\sim}{\nabla} T_s \cdot \underset{\sim}{\nabla} T' - \frac{S}{K} T'\right] dV$$

$$= 2 \int_D \left[\underset{\sim}{\nabla} \cdot \left[T' \nabla T_s\right] \underbrace{- T' \nabla^2 T_s - \frac{S}{K} T'}_{\substack{\parallel \\ -T'\left(\nabla^2 T_s - \frac{S}{K}\right) = 0\,!}}\right] dV$$

$$= 2 \int_{\partial D} \overset{0 \text{ on } \partial D}{\cancel{T'}}\, \underset{\sim}{\nabla} T_s \cdot \underset{\sim}{n}\, dA = \underline{\underline{0}}$$

Thus:

$$I = I_S + \int_D (\underset{\sim}{\nabla} T')^2 \, dV$$

and hence $I$ is a minimum when $\underset{\sim}{\nabla} T' = 0$ or when $T = T_S$

We may <u>choose</u> approx form for $T$ which satisfies B.C.'s, then take derivatives of adjustable parameters & set equal to zero

Will solve for HW

# Conformal Mapping:

Heat cond. in solids can be simplified further if 2-D., allows use of technique of conformal mapping

Described in Churchill: Complex variables & applications. Read Ch. 8&9

Use complex variables to obtain solutions.

Consider any analytic $f^n$ $W(z)$ where $z=x+iy$

(Analytic means derivative exists throughout entire region.

Then $W(z)=u(x,y)+iV(x,y)$ and both u&V are harmonic,

i.e. $\nabla^2 u=\nabla^2 V=0$

provides useful technique to obtain sol'n to 2-D ht. eq'n problems

Since <u>any</u> analytic $f^n$ will give sol'n to eq'n,

easy to find a physical problem corresponding to a solution, but less simple to find a solution corresp. to a set of B.C.'s

To do latter, use conformal mapping.

Let $u+iV=W(x+iy)$ where $W(z)$ is analytic

This represents a conformal mapping:

$u=u(x,y)$, $V=V(x,y)$

Now if $h(u,v)$ is harmonic, i.e. $h_{uu}+h_{vv}=0$ then $H(x,y)=h(u(x,y),v(x,y))$ is <u>also</u> harmonic

$$H_{xx}+H_{yy}=0 \ !$$

This immediately suggests a way to obtain sol'ns for complex 2-D bdy's:

Map complex bdy in x-y plane onto simple bdy in u,v plane, write down sol'n $h(u,v)$ & substitute transform ⇒ two problems:

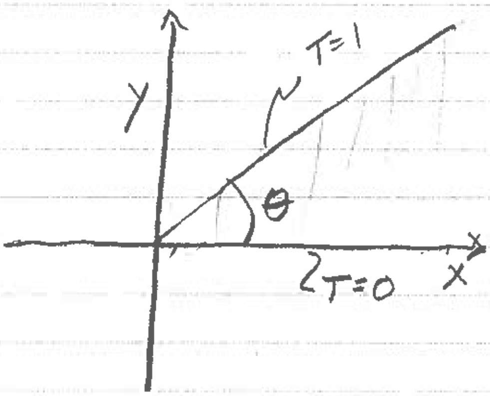
1) obtaining correct transform may be difficult, although many are tabulated ⇒ back of Churchill useful

2) Mapping may alter B.C.'s: only $h=cst$ and $\nabla h\cdot \underset{\sim}{n}=0$ are preserved on bdy, thus if have b.c. $\nabla h\cdot \underset{\sim}{n}=cst$ may obtain simple bdy <u>shape</u>, but have complex b.c.

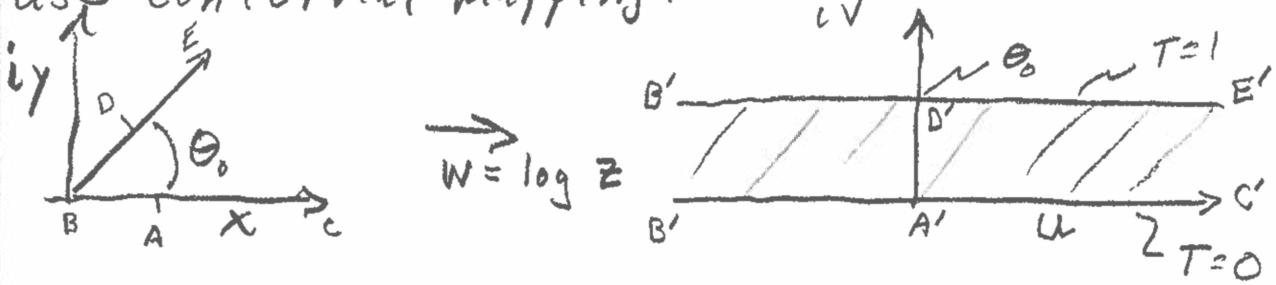
End lec. 23

Work some examples:

Temp. profile in a wedge:



Could solve this using cylindrical coords, instead use conformal mapping:



Thus in the new geometry, the solution is just:

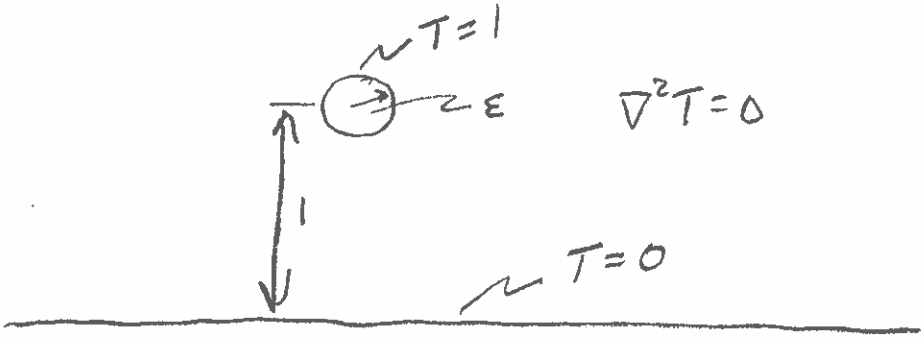
$$T = \frac{v}{\theta_0}$$

where $v = \mathrm{Im}(\log z) = \tan^{-1}\left(\frac{y}{x}\right)$

Thus $T = \dfrac{\tan^{-1}(y/x)}{\theta_0} = \dfrac{\theta}{\theta_0}$

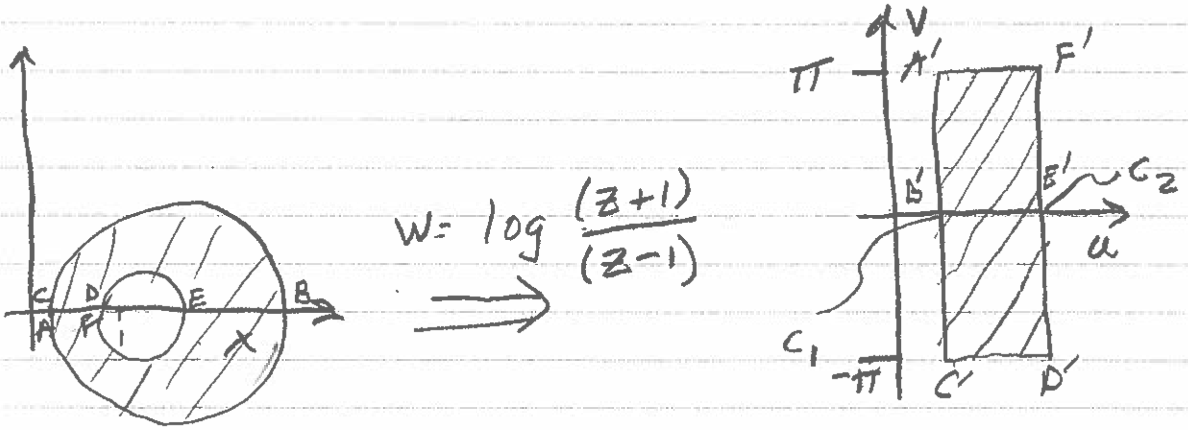
A more complex problem: often transforms that are necessary are not tabulated, so must use pairs of transforms or modify transforms in some way.

Look at 2-D analog to sphere-plane problem:



what is temp. distrib? This cannot be solved using method of reflections since sol'n without plane cannot match B.C. at ∞.

Can't find this geometry in tables, however from Churchill, find (#21):



where centers of circles are at $z = \coth(c_n)$,

radii are $\operatorname{csch}(c_n)$, $n = 1, 2$

This is actually the required transform.

Note that the plane $x = 0$ maps onto $u = 0$,

i.e. plane $x = 0$ is circle of radius $R$ centered at $R$ in limit $R \to \infty$, $\therefore (c_n)_{plane} = 0$

We need to map circle centered at 1 and of radius $\varepsilon$ onto circle of radius $\operatorname{csch}(c_n)$ and centered at $\coth(c_1)$

Use simple scaling:

$$z' = Az$$

where $A = \coth(c_1)$; $A\varepsilon = \operatorname{csch}(c_1)$

Thus

$$A^2 = \frac{\cosh^2 c_1}{\sinh^2 c} = \frac{\sinh^2 c + 1}{\sinh^2 c}$$

with $A^2 \varepsilon^2 = \frac{1}{\sinh^2 c}$

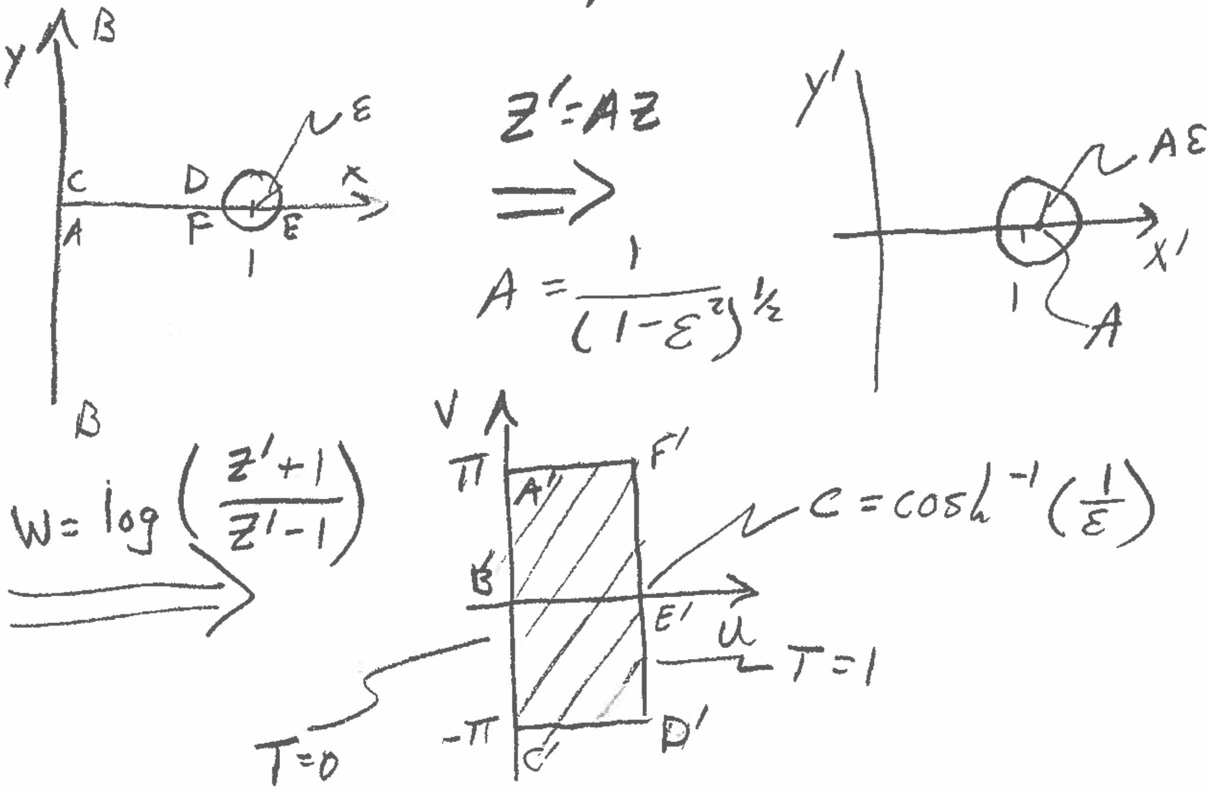
Thus:

$$A^2 = \frac{1 + \frac{1}{A^2\varepsilon^2}}{1/A^2\varepsilon^2} = A^2\varepsilon^2 + 1$$

or $A = \dfrac{1}{(1-\varepsilon^2)^{1/2}}$

and $c = \cosh^{-1}(1/\varepsilon)$

Thus we have the composite transformations:



where $\nabla^2_{u,v} T = 0$

hence $T = \dfrac{u}{c} = \dfrac{u}{\cosh^{-1}(1/\varepsilon)}$

where

$$u = \mathrm{Re}\left\{\log\left(\frac{z'+1}{z'-1}\right)\right\} = \mathrm{Re}\left\{\log\left(\frac{z+(1-\varepsilon^2)^{1/2}}{z-(1-\varepsilon^2)^{1/2}}\right)\right\}$$

We may represent any complex number by its polar form:

$$x + iy = r e^{i\theta}$$

where $r = (x^2+y^2)^{1/2} \equiv (z\bar{z})^{1/2} \equiv \left[(x+iy)(x-iy)\right]^{1/2}$

and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Thus:

$$\mathrm{Log}(z) = \log r + i\theta$$

and $\mathrm{Re}\{\log(z)\} = \log|r|$

hence:

$$u = \frac{1}{2}\log\left\{\frac{\left(z+(1-\varepsilon^2)^{1/2}\right)\left(\bar{z}+(1-\varepsilon^2)^{1/2}\right)}{\left(z-(1-\varepsilon^2)^{1/2}\right)\left(\bar{z}-(1-\varepsilon^2)^{1/2}\right)}\right\}$$

$$= \frac{1}{2}\log\left[\frac{x^2+y^2+1-\varepsilon^2+2x(1-\varepsilon^2)^{1/2}}{x^2+y^2+1-\varepsilon^2-2x(1-\varepsilon^2)^{1/2}}\right]$$

which is a rather simple way to obtain a complex result.

What is the temperature distribution near the cylinder?

Define new coordinates (cylindrical) centered on cylinder:

$x = 1 + \varepsilon r\cos\theta \,,\; y = \varepsilon r\sin\theta$

∴ cylinder ≡ $r = 1$

$$\therefore T = \frac{\log\left[\dfrac{2(1+\varepsilon r\cos\theta)(1+(1-\varepsilon^2)^{1/2}) + \varepsilon^2(r^2-1)}{2(1+\varepsilon r\cos\theta)(1-(1-\varepsilon^2)^{1/2}) + \varepsilon^2(r^2-1)}\right]}{\log\left[\dfrac{1+(1-\varepsilon^2)^{1/2}}{1-(1-\varepsilon^2)^{1/2}}\right]}$$

As $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0}\left(1-(1-\varepsilon^2)^{1/2}\right) = \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^4)$$

$$\therefore T \simeq \frac{\log\left[\dfrac{4}{\varepsilon^2(1+r^2-1)}\right]}{\log\left[\dfrac{4}{\varepsilon^2}\right]}$$

$$= \frac{\log(2/\varepsilon r)}{\log(2/\varepsilon)}$$ where $\varepsilon \ll 1$; $1 < r \ll \frac{1}{\varepsilon}$

~~which blows up for $\varepsilon = 0$ as expected~~

$$= \frac{\log(2/\varepsilon) - \log(r)}{\log(2/\varepsilon)} = 1 - \frac{\log(r)}{\log(2/\varepsilon)} \rightarrow 1$$

as $\varepsilon \to 0$

(uniform temp)