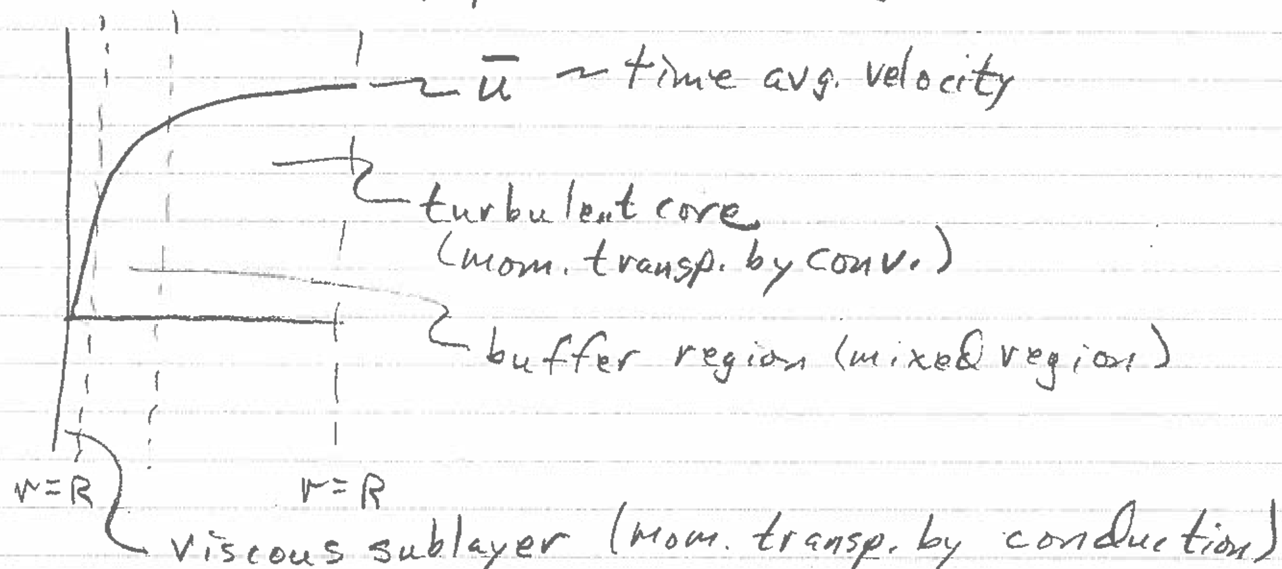
Turbulence

Thus far, have considered only laminar flow condit. thus restricted to laminar convection & molecular thermal diffusion. Now look at effect of turbulence.

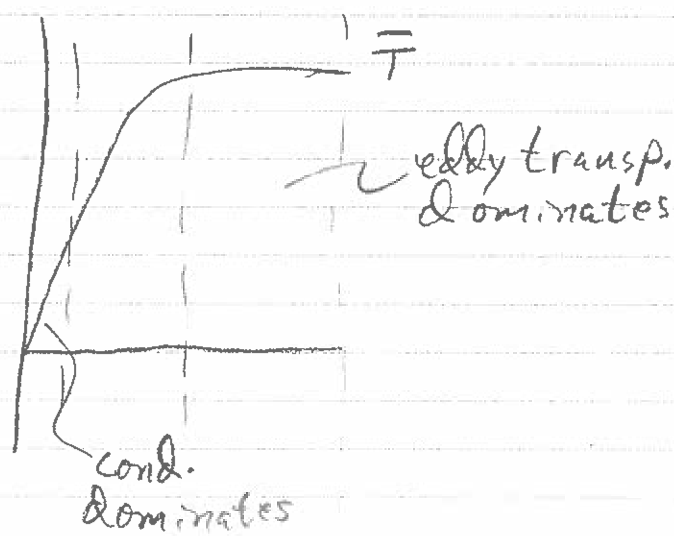
In laminar system, only way energy can cross streamline is by diffusion.

In turbulent system, may have convective transp. of E across avg. streamlines: in highly turbulent systems, conv. transp >> diffusion

Look at flow in pipe: three regions



(qualitatively)
Same diagram for energy transp:



Averaged quantities defined:

Let temp. be given by

$T = \bar{T} + T'$ → rnd fluctuations (due to eddy transp)

↳ time avg.

In turb. system, $T' \sim 5$-10% of ΔT in pipe

$$\bar{T} = \frac{1}{\tau}\int_0^{\tau} T\,dt = \frac{1}{\tau}\int_0^{\tau} (\bar{T} + T')\,dt$$

$$\bar{T} \neq f^n(t) \therefore \bar{T} = \bar{T} + \frac{1}{\tau}\int_0^{\tau} T'\,dt$$

$\therefore \overline{T'} = 0$ by definition

similarly define:

$u_x = \bar{u}_x + u_x'$, $p = \bar{p} + p'$, etc.

Note that while $\overline{A'} = 0$, $\overline{(A'^2)} \neq 0$ ⇒ gives meas. of rnd. fluctuations in A

Also $\overline{AB} = \bar{A}\,\bar{B} + \overline{A'B'}$

last term may be non-zero if A', B' are correlated

expect u_x', T' to be correlated.

End lec 24

Let's look at turbulent ht. transf. in a pipe.

How can we solve this? Build a pipe & meas. ht. transf. is best way (most accurate at least).

Can't solve exactly: write down eq'ns (still continuum as smallest eddies >> mean free path) but obtain full 3-D non-linear time dep. mess. Turb. considered (perhaps) greatest unsolved problem in physics.

Approximate: use time averaged approach

molecular shear stress

Eq'n of motion:

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} = \underset{\sim}{\nabla} \cdot \underset{\approx}{\tau}^{l} - \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u} \underset{\sim}{u}) - \underset{\sim}{\nabla} P + \rho \underset{\sim}{g}$$

or, in Einstein notation: $\tau_{ij}^{l} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$

$$\rho \frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x_j} \left(\tau_{ij}^{l} - \rho u_i u_j \right) - \frac{\partial P}{\partial x_j} + \rho g_i$$

Taking average value: $= \mu \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right)$

$$\rho \frac{\partial \overline{u}_i}{\partial t} = \frac{\partial}{\partial x_j} \left(\overline{\tau_{ij}^{l}} - \rho \overline{u_i u_j} \right) - \frac{\partial \overline{P}}{\partial x_j} + \rho g_i$$

look at term $\overline{u_i u_j} = \overline{u}_i \overline{u}_j + \overline{u_i' u_j'}$

$$\therefore \rho \frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = - \frac{\partial \overline{P}}{\partial x_j} + \frac{\partial \overline{\tau_{ij}^{l}}}{\partial x_j} - \left[\frac{\partial}{\partial x_j} \left(\rho \overline{u_i' u_j'} \right) \right] + \rho g_i$$

by time averaging, wind up w/ extra term due to non-linearity

$-\rho \overline{u_i' u_j'}$ has form of stress, so called

$\overline{\tau}_{ij}^{(t)} \equiv$ Reynolds stress

Let $\overline{\tau_{ij}} \equiv \tau_{ij}^{(\ell)} + \tau_{ij}^{(t)}$.

So:

$$\rho \frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial \overline{\tau_{ij}}}{\partial x_j} + \rho g_i$$

Which is the same eq'n used in laminar flow but

$$\overline{\tau_{ij}} \neq \mu \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right)$$

so system of equations for $\bar{u}_i$ is not closed

Have to assume some form for $\tau_{ij}^{(t)}$ to close system.

For isotropic turbulence, usually assume:

$$\tau_{ij}^{(t)} = \mu^{(t)} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right)$$

where $\mu^{(t)} \equiv$ eddy viscosity: $f^{\underline{n}}$ of flow $\therefore$ strong $f^{\underline{n}}$ of position.

If turbulence is anisotropic, $\mu^{(t)}$ will be a tensor:

$$\tau_{ij}^{(t)} = \mu_{ijk\ell}^{(t)} \left(\frac{\partial \bar{u}_k}{\partial x_\ell} + \frac{\partial \bar{u}_\ell}{\partial x_k} \right)$$

Look at eq'n of energy:

$$\frac{\partial}{\partial t}(\rho \hat{C}_p T) + \underset{\sim}{\nabla} \cdot (\rho \hat{C}_p \underset{\sim}{u} T) = -\underset{\sim}{\nabla} \cdot \underset{\sim}{q} + \underset{\approx}{\tau} : \underset{\sim}{\nabla} \underset{\sim}{u}$$

or

$$\frac{\partial}{\partial t}(\rho \hat{C}_p T) + \frac{\partial}{\partial x_j}(\rho \hat{C}_p u_j T) = -\frac{\partial q_j}{\partial x_j} + \tau_{ij} \frac{\partial u_i}{\partial x_j}$$

Taking average as before:

$$\frac{\partial}{\partial t}\left(\rho \hat{C}_p \bar{T}\right) + \frac{\partial}{\partial x_j}\left(\rho \hat{C}_p \bar{u}_j \bar{T}\right) = -\frac{\partial}{\partial x_j}\left(\bar{q}_j^{(l)} + \rho \hat{C}_p \overline{u_j' T'}\right.$$

$$+ \mu \overline{\phi_v^{l}} + \mu \overline{\phi_v^{t}}$$

Note similarity to averaged mom. eq'n: covariance of u_j' & T' gives term analogous to conduction:

$\bar{\underset{\sim}{q}} = \bar{\underset{\sim}{q}}^{(l)} + \bar{\underset{\sim}{q}}^{(t)}$ where $\bar{\underset{\sim}{q}}^{(t)} = \rho \hat{C}_p \overline{\underset{\sim}{u}' T'}$:

add'l diffusion arising from rad. motion of packets of fluid.

Recall that for isotropic mat'l, $\underset{\sim}{q} = -k \underset{\sim}{\nabla} T$

$\therefore$ let $\bar{\underset{\sim}{q}}^{(t)} = -k^{(t)} \underset{\sim}{\nabla} \bar{T}$ for isotropic turbulence

If anisotropic, then $k^{(t)} \equiv$ tensor, i.e,

$$\bar{q}_i^{(t)} = -K_{ij}^{(t)} \frac{\partial \bar{T}}{\partial x_j}$$

Now look at $\overline{\phi_v^{(l)}}$, $\overline{\phi_v^{(t)}}$:

In Einstein notation $\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

for an incompressible fluid.

Thus $\phi_v = \frac{1}{\mu} \overline{\tau_{ij} \frac{\partial u_i}{\partial x_j}} = \overline{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}\right)}$

$$= \left(\frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_j}{\partial x_i} \right) + \left(\overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}} + \overline{\frac{\partial u_j'}{\partial x_i} \frac{\partial u_i'}{\partial x_j}} \right)$$

$$= \bar{\phi}_V^{(l)} + \bar{\phi}_V^{(t)}$$

Thus, if ϕ_V is important, must determine some relation between $\bar{\phi}_V^{(t)}$ & $\bar{u}_i$ as well.

Thus we have equations:

$$\rho \frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[(\mu + \mu^{(t)}) \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \right] + \rho g_i$$

and (neglecting viscous dissipation):

$$\frac{\partial}{\partial t}(\rho \hat{C}_p \bar{T}) + \frac{\partial}{\partial x_j}(\rho \hat{C}_p \bar{u}_j \bar{T}) = \frac{\partial}{\partial x_j}\left[(k + k^{(t)}) \frac{\partial \bar{T}}{\partial x_j} \right]$$

where $\mu^{(t)}$, $k^{(t)}$ are functions of $\bar{u}_i$, geometry

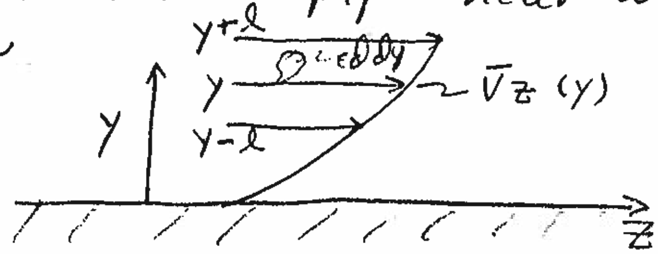
If we have some model for $\mu^{(t)}$, $k^{(t)}$ can solve above eq'ns for averaged quantities.

Prandtl's mixing length theory:

eddies are responsible for momentum as well as energy transport.

Consider flow thru pipe near wall:

$y = R - r$



Think of eddy as a packet of momentum and energy, transports ρ & T as moves across streamlines analogous to molecule in kinetic theory of gases.

Imagine eddy at $y+l$, moves toward wall with velocity v_y' & breaks up (deposits momentum, merges w/ fluid) after distance $l \Rightarrow$ char. length scale. Look at momentum flux:

Thus, transp. of z-mom. from this eddy given by:

$$\frac{\text{z mom. transp.}}{\text{vol. of eddy}} = \rho \Delta \bar{V}_z = \rho\left(\bar{V}_z\Big|_{y+l} - \bar{V}_z\Big|_y\right)$$

and from eddy transp. from $y-l$: (also moves w/ velocity v_y')

$$\frac{\text{z-mom. transp.}}{\text{vol of eddy}} \propto -\rho\left(\bar{V}_z\Big|_y - \bar{V}_z\Big|_{y-l}\right)$$

Thus momentum flux (see Ch. 1–4 BSL)/unit vol. of eddies is

$$\frac{\rho}{2}\left(\bar{V}_z\Big|_{y+l} - \bar{V}_z\Big|_{y-l}\right)$$

Now include <u>rate</u> of eddies moving across plane y:

$$\tau_{yz}^{(t)} \cdot A_y = \frac{1}{2}\left(\rho \bar{V}_z\Big|_{y+l} - \rho \bar{V}_z\Big|_{y-l}\right) * |v_y'|\, A_y$$

$$\cong \tau_{yz}^{(t)} = |v_y'|\,\rho\, l\, \frac{d\bar{V}_z}{dy} \quad \text{if } \bar{V}_z \text{ is linear over } l$$

However, l <u>may</u> approach the dimensions of the duct for large eddies, so this may not be true always.

Let's take lateral velocity of eddy as lateral r.m.s. fluctuating velocity, i.e. let

$$\overline{|V_y'|} = \left(\overline{V_r'^2}\right)^{1/2}$$

Thus, get $\quad \dfrac{\mu^{(t)}}{\rho} = l\left(\overline{V_r'^2}\right)^{1/2} \quad$ from $\tau_{yz}^{(t)} = -\mu^{(t)}\dfrac{d\bar{V}_z}{dy}$

Now for second part: what is $\left(\overline{V_r'^2}\right)^{1/2}$?

Let's assume $\quad \underset{\text{lateral}}{\left(\overline{V_r'^2}\right)^{1/2}} = \underset{\text{axial}}{\left(\overline{V_z'^2}\right)^{1/2}}$

would be true if isotropic, but rather unlikely here. As an eddy moves laterally, it carries z-mom, so it changes velocity. If this is source of V_z', find that

$$\left(V_z'^2\right)^{1/2} = \left|\frac{d\bar{V}_z}{dr}\right| l$$

and $\quad \dfrac{\mu^{(t)}}{\rho} = l^2\left|\dfrac{d\bar{V}_z}{dr}\right|$

Still must determine l:

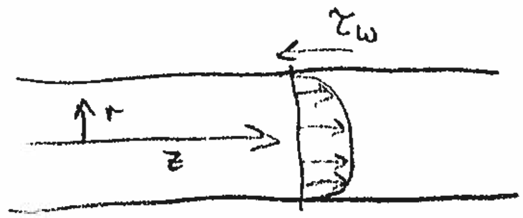
Since eddies are largest in core & smallest near walls, Prantl assumed that

$$l = \underbrace{K}_{\text{von Karman const., has exp'tl value} \sim 0.4 \Rightarrow \text{nicely } O(1)} y$$

And $\quad \dfrac{\mu^{(t)}}{\rho} = K^2 y^2 \left|\dfrac{d\bar{V}_z}{dr}\right|$

which can be used to determine $\bar{V}_z$ (closes eq'ns)

What is the magnitude of $\mu^{(t)}/\mu$? Consider fully dev. turbulent pipe flow:



$\bar{V}_z \neq f^{n}(z)$

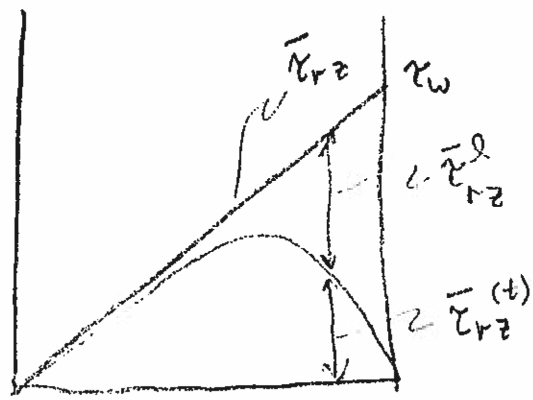
total:

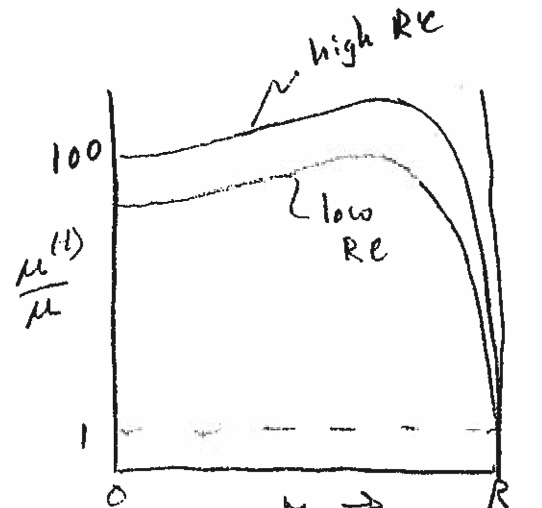
get $-\dfrac{d(r\bar{\tau}_{rz})}{r\,dr} = -\dfrac{\Delta P}{L} = -\bar{\Phi}_M = -\dfrac{2\tau_w}{R}$

force balance

Thus $\bar{\tau}_{rz} = \dfrac{r}{R}\tau_w \Rightarrow$ linear increase

$= \bar{\tau}_{rz}^{(l)} + \tau_{rz}^{(t)}$





$\rightarrow -\mu\dfrac{d\bar{V}_z}{dr} - \mu^{(t)}\dfrac{d\bar{V}_z}{dr}$ where $\mu^{(t)} = \rho K^2 y^2 \left|\dfrac{dV_z}{dr}\right|$

End lec. 25

Now $\tau_{rz}^{(l)} = 0$ at $r=0$, max at wall

$\tau_{rz}^{(t)} = 0$ at $r=0$ → no shear at center, $=0$ at wall (no turbulence)

To obtain $\bar{V}_z(r)$, must integrate:

$$-\mu\frac{d\bar{V}_z}{dr} - \rho K^2 (R-r)^2 \left|\frac{d\bar{V}_z}{dr}\right| \frac{d\bar{V}_z}{dr} = \tau_w \frac{r}{R} = \frac{\Delta P r}{2L}$$

Now $\dfrac{d\bar{V}_z}{dr} < 0$ (flow highest in middle - $r=0$)

thus $\left(\dfrac{d\bar{V}_z}{dr}\right)^2 (R-r)^2 \rho K^2 - \left(\dfrac{d\bar{V}_z}{dr}\right)\mu - \dfrac{\Delta P r}{2L} = 0$

Simply quadratic eq'n for $\frac{\partial \bar{V}_z}{\partial r}$, thus:

$$\frac{\partial \bar{V}_z}{\partial r} = \frac{\mu - \sqrt{\mu^2 + 2\frac{\Delta P r}{L}(R-r)^2 \rho K^2}}{2(R-r)^2 \rho K^2}$$

Can integrate to get $\bar{V}_z$, i.e.

$$\bar{V}_z = -\int_r^R \frac{\partial \bar{V}_z}{\partial r'}\, \partial r'$$

Now $\mu^{(t)} = \rho K^2 (R-r)^2 \left|\frac{\partial V_z}{\partial r}\right|$

$$\therefore \frac{\mu^{(t)}}{\mu} = \frac{1}{2}\left(-1 + \sqrt{1 + \frac{r}{R}\left(1-\frac{r}{R}\right)^2 2\frac{\Delta P R^3 \rho K^2}{L\mu^2}}\right)$$

$$\therefore \frac{\mu^{(t)}}{\mu} = f^{n}\left(2\frac{\Delta P R^3}{L}\frac{\rho K^2}{\mu^2}\right) \equiv f^{n}\left(\left(\frac{\Delta P R^2}{L\mu}\right)\frac{2R\rho}{\mu}K^2\right)$$

char. velocity

$$\equiv f^{n}(Re \cdot K^2)$$

as $Re \to 0$ or $1-\frac{r}{R} \to 0$, $\mu^{(t)} \to 0$

Also gives result <u>that</u> $\mu^{(t)} \to 0$ at centerline: <u>not</u> correct as $\overline{V_r' V_r'} \neq 0$ at center, just diminished.

In order to look at flow in core, must modify analysis: now it gets <u>really</u> approximate

Near core, assume viscous forces unimp.

$\therefore$ obtain balance:

$$\tau_w \frac{r}{R} \equiv \tau_w\left(1-\frac{y}{R}\right) = \tau_{rz}^{(t)} \equiv \rho K^2 y^2 \left(\frac{\partial \bar{V}_z}{\partial y}\right)^2$$

Here Prandtl makes assumption that $1-\frac{y}{R} \approx 1$!

$$So, \quad \frac{\partial \bar{V}_z}{\partial y} = \frac{1}{K}\sqrt{\frac{\tau_w}{\rho}}\frac{1}{y} \equiv \frac{1}{K} V_* \frac{1}{y}$$

char. velocity

for region outside of buffer region $\Rightarrow$ still near wall?

We can integrate, but rather than from wall, integ from edge of buffer region. (at y_1) w/ velocity $\bar{v}_{z1}$:

$$\frac{\bar{v}_z}{v_*} - \frac{\bar{v}_{z1}}{v_*} = \frac{1}{\kappa} \ln \frac{y}{y_1}$$

We have viscous length scale $\frac{\mu}{\rho v_*}$

$\therefore$ Define $y^+ \equiv \frac{y v_* \rho}{\mu}$, $v_z^+ = \frac{\bar{v}_z}{v_*}$,

thus $v_z^+ - v_{z1}^+ = \frac{1}{\kappa} \ln \frac{y^+}{y_1^+}$

Now turn to exp't :

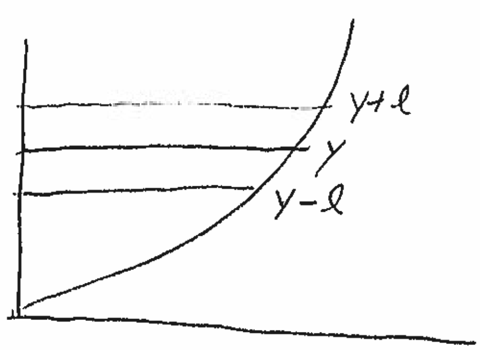
for Re > 20,000, have $\kappa \approx 0.36$, $y_1^+ = 26$, $\bar{v}_z^+ = 12.85$

thus $v_z^+ \approx \frac{1}{0.36} \ln y^+ + 3.8$

for $y^+ > 26$

Done w/ pure fluid,

Now look at analogous Dev. for <u>Temp</u>



$$\frac{\text{energy carried}}{\text{vol. of eddy}} = \rho \hat{C}_p \bar{T}_{y+l} \text{ \& } \rho \hat{C}_p \bar{T}_{y-l}$$

$$\therefore \frac{\text{transport}}{\text{vol of eddy}} = \frac{\rho}{2} \hat{C}_p \frac{\partial \bar{T}}{\partial y} \cdot 2l$$ (assume linear over $\Delta y = 2l$)

Thus flux of energy transp. from this mechanism:

$$q_y^{(t)} = \Big(\underbrace{\rho \hat{C}_p \frac{\partial \bar{T}}{\partial y}\, l}_{\text{energy transferred}}\Big) \cdot \underbrace{|v_y'|}_{\text{velocity}} = \rho \hat{C}_p\, l^2 \frac{d\bar{T}}{dy}\left|\frac{d\bar{v}_z}{dr}\right|$$

where we have substituted $l\left|\frac{d\bar{v}_z}{dr}\right|$ for $|v_y'|$

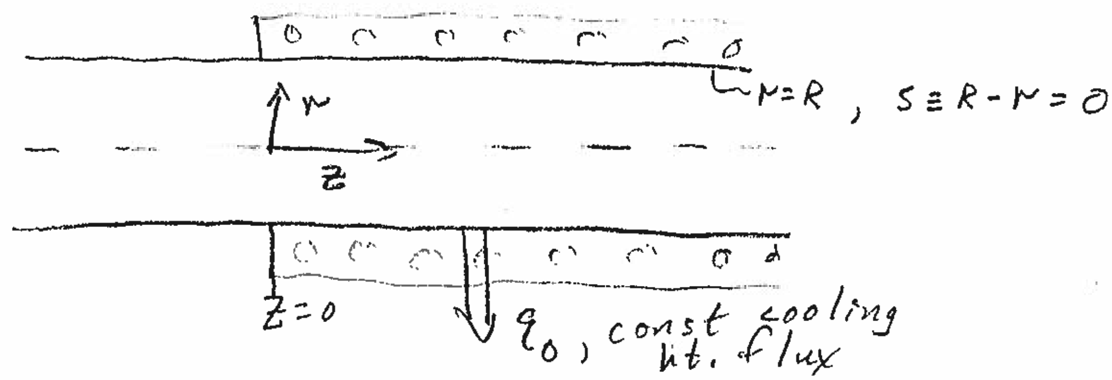
Thus $k^{(t)} = \rho \hat{C}_p\, l^2 \left|\frac{d\bar{v}_z}{dr}\right|$

Look at Prandtl #:

$$Pr^{(t)} = \frac{\mu^{(t)}/\rho}{k^{(t)}/\rho \hat{C}_p} = \frac{\nu^{(t)}}{\alpha^{(t)}} = 1 \Rightarrow$$ same mechanism for momentum & energy transp.

To see how all this works, look at example:

12.3-1 BS&L: Turb. flow in pipe w/ const ht. flux at walls.



Obtain equations which govern ave ht. transf.:

(neglect viscous dissip.) $\quad\to 0\quad$ (steady state)

$$\rho \hat{C}_p\left[\cancel{\bar{v}_r \frac{\partial \bar{T}}{\partial r}} + \bar{v}_\theta \frac{\partial \bar{T}}{\partial \theta} + \bar{v}_z \frac{\partial \bar{T}}{\partial z}\right]$$

$\to 0$ by symmetry

$$= -\left[\frac{1}{r}\frac{\partial}{\partial r} r\left(\bar{q}_r^{\,l} + \bar{q}_r^{\,t}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\bar{q}_\theta^{\,l} + \bar{q}_\theta^{\,t}\right) + \frac{\partial}{\partial z}\left(\bar{q}_z^{\,l} + \bar{q}_z^{\,t}\right)\right]$$

Last term also disappears: Axial "conduction" by molecular conduction & eddy transp. ⇒ small w.r.t. convection.

Alternative, look at large z:

$$\bar{T} \sim Az + \psi(r)$$

$$\therefore \bar{q}_z^{\,l} \ \& \ \bar{q}_z^{\,t} \sim A \cdot f^n(r), \ \therefore \ \frac{\partial \bar{q}}{\partial z} = 0$$

Solve for temp distrib. at large z:

$$\bar{T}(r,z) = Az + \psi(r)$$

$$\therefore \ \rho \hat{C}_p \bar{V}_z A = -\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\bar{q}_r^{\,l} + \bar{q}_r^{\,t}\right)\right]$$

May integrate:

$$\int_r^R \rho \hat{C}_p \bar{V}_z A \, r \, dr = -\left[r\left(\bar{q}_r^{\,l} + \bar{q}_r^{\,t}\right)\right]_r^R$$

$$= -Rq_0 + r\left(\bar{q}_r^{\,l} + \bar{q}_r^{\,t}\right)$$

↳ at outer wall

$$\text{or} \quad \bar{q}_r^{\,l} + \bar{q}_r^{\,t} - \frac{R}{r} q_0 = \frac{1}{r}\int_r^R \rho \hat{C}_p \bar{V}_z A \, r \, dr$$

<u>In the Core</u>: As w/ momentum flux, where we assumed $\tau_{rz} = \tau_w = \underline{cst}$, so assume ht. flux is constant, i.e., set $R/r = 1$, throw out RHS & laminar terms ⇒ very questionable

Thus $\bar{q}_r^{\,t} = q_0$

and $q_0 = \rho \hat{C}_p K^2 s^2 \left(\frac{d\bar{V}_z}{ds}\right)\frac{d\bar{T}}{ds}$

$s = R - r$

Recall that for momentum transport, obtain

$$\tau_w = \rho K^2 s^2 \left(\frac{d\bar{V}_z}{ds}\right)^2$$

Dividing, obtain:

$$\frac{q_o}{\tau_w} = \frac{\hat{C}_p \, d\bar{T}}{d\bar{V}_z}$$

which may be integrated from some s_1:

$$\frac{\hat{C}_p(\bar{T} - T_1)}{q_o} = \frac{\bar{V}_z - \bar{V}_{z,1}}{\tau_o},$$

i.e. the velocity & temp. distributions in the core are similar, (not surprising as same transport mechanism for both) which agrees w/ exp't.

As in mom transfer, appropriate surface s_1 is edge of buffer zone,

Define dim. velocity $V^+ = \bar{V}_z / V_*$ where $V_* = \sqrt{\tau_w/\rho}$,

$T^+ = \dfrac{\rho \hat{C}_p V_* (\bar{T} - T_0)}{q_o}$, $s^+ = \dfrac{s V_* \rho}{\mu}$

T_0 = temp. at wall ($f^{n}(z)$)

Thus $T^+ - T_1^+ = V^+ - V_1^+$

and from earlier sol'n: $T^+ - T_1^+ = \frac{1}{K_1} \ln \frac{s^+}{s_1^+}$,

$\underline{s^+ \geq 26}$

<u>b.</u> Region near wall: Can no longer neglect $\bar{q}_r^{l}$, but will still assume that $q \equiv q_o$, $r/R \approx 1$

$$\therefore \bar{q}_r^{(l)} + \bar{q}_r^{(t)} = q_o$$

Have Fourier law of ht. cond. for $\bar{q}_r^{l}$, must use some expression for $\bar{q}^t$: Rather than Prandtl theory which is not good at walls, use Deissler's Empirical formula:

$$q_r^t = + \rho \hat{C}_p n^2 \bar{V}_z s \left(1 - \exp\left(-n^2 \bar{V}_z s/\nu\right)\right) \frac{d\bar{T}}{ds}$$

where $n = 0.124$

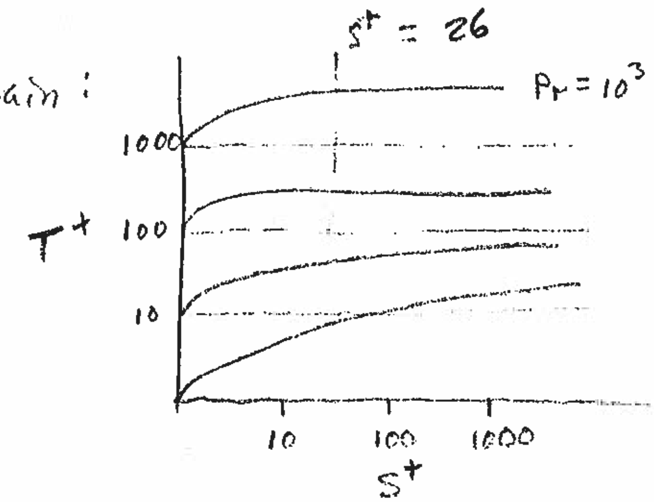
Thus $$q_0 = k \frac{d\bar{T}}{ds} + \rho \hat{C}_p n^2 \bar{V}_z s \left(1 - \exp\left(-n^2 \bar{V}_z s/\nu\right)\right) \cdot \frac{d\bar{T}}{ds}$$

May integrate to obtain $\bar{T}(s)$:

$$T^+ = \int_0^{s^+} \frac{ds^+}{(1/Pr) + n^2 V^+ s^+ \left(1 - \exp(-n^2 V^+ s^+)\right)}$$

where V^+ is determined in a similar manner

Obtain:



Similarity between mechanisms of ht. & momentum transp. leads to Reynolds analogy, i.e.

See ch. 5 & 12 for details
Read Ch 12 &
Do problems 12.B & 12.C

Recall that $$\frac{\hat{C}_p(\bar{T} - \bar{T}_1)}{q_0} = \frac{\bar{V}_z - \bar{V}_{z1}}{\tau_0}$$

where (1) was some ref. position in turbulent core. If we (without justification) let (1) be the wall, multiply by $\rho \bar{V}_z$ & average over the tube, then:

$$\frac{\rho \hat{C}_p \langle \bar{V}_z \rangle \left[\frac{\langle \bar{V}_z \bar{T} \rangle}{\langle \bar{V}_z \rangle} - \bar{T}_0 \right]}{q_0} = \frac{\rho \langle \bar{V}_z^2 \rangle}{\tau_0}$$

if $\langle \bar{V}_z^2 \rangle \approx \langle \bar{V}_z \rangle^2$ (not bad in tubes as flow profile is nearly flat anyway)

and recognize $\bar{T}_b = \dfrac{\langle \bar{V}_z \bar{T} \rangle}{\langle \bar{V}_z \rangle}$,

Thus if $h \equiv \dfrac{q}{T_b - T_0}$, we obtain

$$\frac{h}{\rho \hat{C}_p \langle \bar{V}_z \rangle} = \underbrace{\frac{f}{2}}_{\text{friction factor}} \quad \text{or} \quad \frac{Nu}{Re\,Pr} = \frac{f}{2}, \quad Re \gg 10^4$$

Works pretty well for $Pr = 1$

for $Pr \neq 1$, obtain experimentally that

$$j_H \equiv \frac{Nu_{(ln)}}{Re\,Pr^{1/3}} = \frac{f}{2} \quad \text{for } Re \gg 10^4$$

$\hookrightarrow$ Colburn analogy, good for smooth pipes,

Discussed in Ch 13, End of discussion of turbulence.

End lec. 26

Radiation

Thus far, looked at conductive & conv. transp. — now look at radiative transp:

Energy transmission via electromagnetic radiation propagating w/ velocity c across space which **may** be a vacuum. Will find that in some cases is surprisingly large source of E transf.

Not course in quantum mechanics, so won't deal w/ complex forms — laser, etc. : Use result of physics to compute ht. transfer

BS&L ch 14:

What is electromagnetic rad?

When molecule is heated, moves into excited state. At low temp, just rotation, higher temp is vibration, plasma temp: ionization

Transition to higher energy state accomplished thermally or by absorption of rad., sim. transition to lower E state by thermal proc. or by emission source of radiation

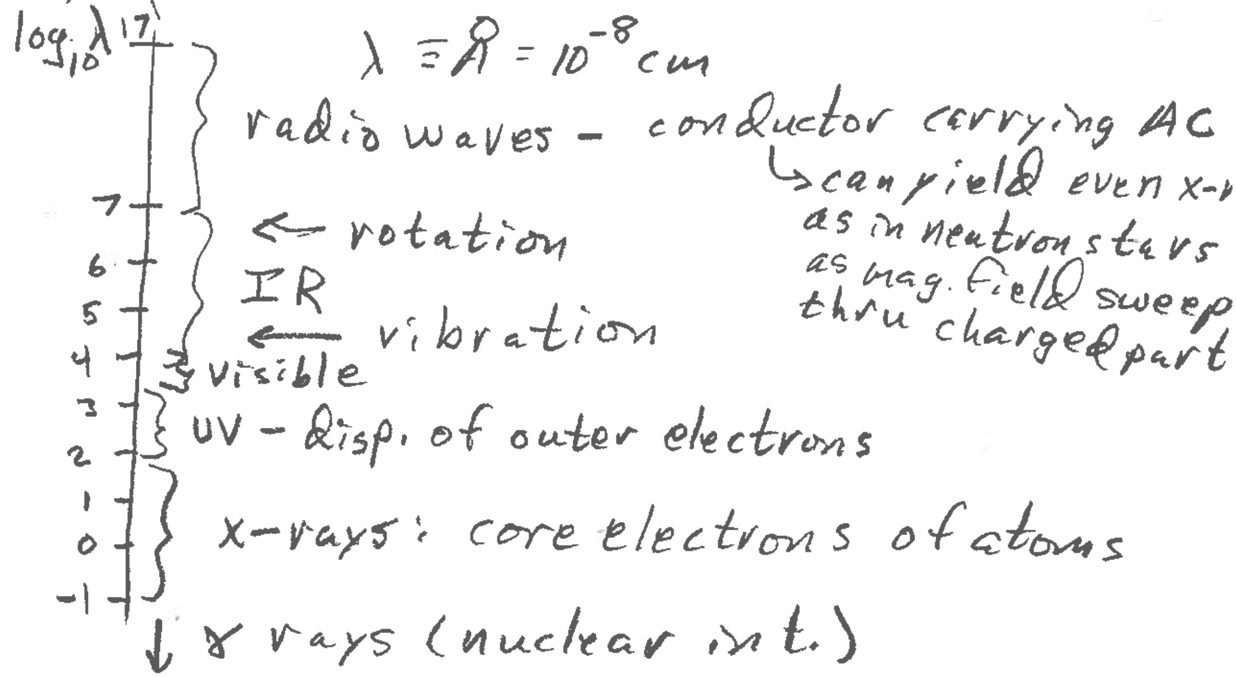
Radiation is quantized ∴ transition corresp. to change in ΔE emits photon w/ energy $\Delta E = h\nu$

$h \Rightarrow$ plank's cst $\nu \Rightarrow$ frequency

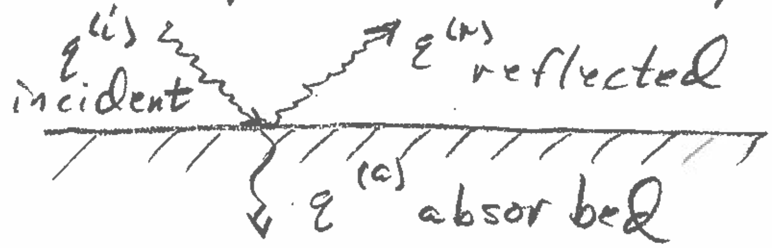
Thus energy level of transition assoc. w/ radiation of particular ν

Have wavelength $\lambda = c/\nu$ $c \equiv 3\times 10^{10}$ cm/sec

EM rad found in continuous spectrum of energy levels:



To study rad. energy transp, need to develop new concepts. Consider opaque solids:



Define absorptivity $a \equiv \dfrac{q^{(a)}}{q^{(i)}}$

In general, will be f$^{\underline{n}}$ of ν & angle of incidence

Have condition for all ν: $a(\nu) \leq 1$

Introduce <u>ideal</u> body:

gray body s.t. $a_\nu < 1$ but <u>indep</u> of ν at all T

if $a_\nu = 1$ then have Black Body

Surface will emit radiation as well

Let flux be given by $q^{(e)}$ & at ν by $q_\nu^{(e)} d\nu$

Black body (as we shall show) emits largest flux of thermal rad:

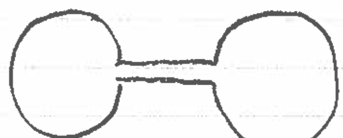
$$q_b^{(e)} \quad \& \quad q_{b\nu}^{(e)} d\nu$$

$\therefore$ define emissivity $e \equiv \dfrac{q^{(e)}}{q_b^{(e)}}$ & $e_\nu = \dfrac{q_\nu^{(e)} d\nu}{q_{\nu b}^{(e)} d\nu} \le$

Now demonstrate that $e = a$!

Think of a cavity: evacuated enclosure w/ isothermal walls. At equil, no net exchange of E between cavity & walls (there will be some energy density in vacuum due to radiation). Rad. in cavity indep of nature of walls $\Rightarrow f^n(T)$ alone!

May prove by simple thought exp't: connec 2 cavities of diff. mat'l but same T:



Energy would flow if not same radiation in each cavity: violates 2nd law.

Rad. similarly uniform & unpolarized throughout cavity:

Isotropic : similarly, universe bathed in isotropic radiation corresp. to few °K.

Cavity characterized by some rad. intensity $q^{(cav)}$ → energy impinging on a solid surface of unit area anywhere in cavity.

Now put small black body in cavity, same temp. as walls. No net interchange of E.

$$\therefore \; q^{(cav)} = \underbrace{q_b^{(a)}}_{a=1} = \underbrace{q_b^{(e)}}_{e=1} \Rightarrow \text{radiation emitted}$$

by black body is identical to equilibrium cavity radiation at same temp.

Now put gray body of same temp into cavity:

$$\text{energy abs} = a\,q^{(cav)} = q^{(e)}$$

$$\text{or } a = \frac{q^{(e)}}{q_b^{(e)}} = e$$

$\therefore$ obtain Kirchoff's law: $a = e$

similarly, can show $a_\nu = e_\nu$

For a black body, can derive a relation between temp. & rad. intensity:

Consider cavity radiation to be a gas made up of photons, each w/ energy $h\nu$ & mom. $h\nu/c$. By consequence of isotropy, energy density given by:

$$u^{(r)} = \frac{4}{c} q_b^{(e)} \Rightarrow E/\text{volume}$$

And momentum of photons exerts pressure on walls:

$$p^{(r)} = \frac{1}{3} u^{(r)}$$

Note: This pressure is real: SF afficianados will recognize that photon pressure is what makes light sail work.

Internal energy of gas $U = V u^{(r)}$, thus from thermo:

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

$$\text{or } u^{(r)} = \frac{T}{3}\frac{\partial u^{(r)}}{\partial T} - \frac{u^{(r)}}{3}$$

$$\equiv \frac{\partial \ln u^{(r)}}{\partial \ln T} = 4 \quad \therefore u^{(r)} = bT^4$$

or $q_b^{(e)} = \sigma T^4 \Rightarrow$ Stefan-Boltzmann law, derived experimentally.

$\sigma = 5.67 \times 10^{-8} \frac{W}{m^2\,°K^4}$

May also obtain from quantum theory: if photons obey Bose-Einstein statistics, then:

$$q^{(e)}_{b\lambda} = \frac{2\pi c^2 h}{\lambda^5} \frac{1}{e^{ch/\lambda kT} - 1} \quad \Big\} \text{ Plank's distri law}$$

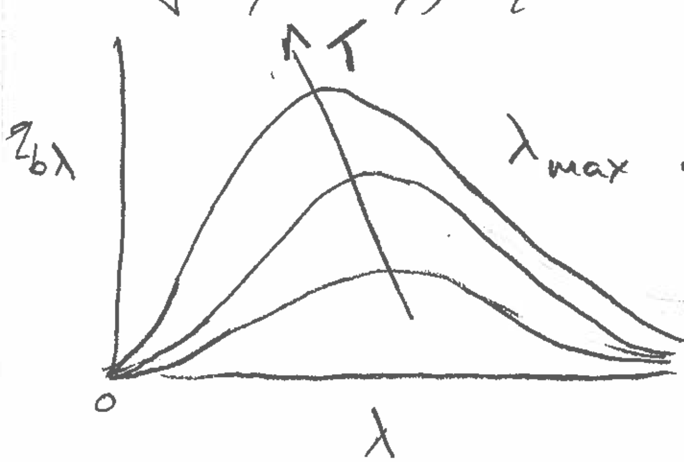
May integrate over λ to obtain:

$$q^{(e)}_{b} = \left(\frac{2}{15} \frac{\pi^5 k^4}{c^2 h^3}\right) T^4$$

$\to \sigma \Rightarrow$ about 1½% below that meas. exp'tally but considered more reliable.

Theory is <u>exact</u> at equilibrium.

For a gray body, $q^{(e)} = e\sigma T^4$

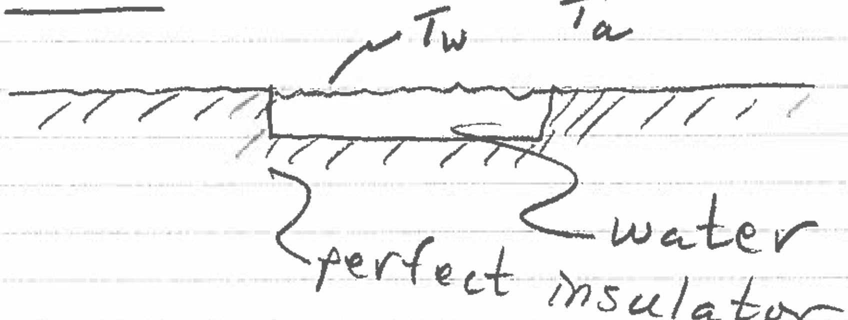


λ_{max} shifts as $f^n(T)$

For the sun, λ_{max} is visible (green) $\Rightarrow$ ~~why plants~~ are green ~~(to absorb max radiation)~~

How can we use this? Ex. 14.5-3, BS&L: At what ambient temp. will a pan of water freeze?

If neglect radiation, answer is 32°F. If in a desert at night, however, can be <u>much</u> warmer



If air is dry & still, only ht transfer (other than radiation) will be free conv.

BS&L gives $q = h(T_a - T_W)$, $h = 0.2(T_a - T_W)^{1}$

where $[h] \equiv \frac{Btu}{hr\, ft^2\, °F}$ & $[T] \equiv °R$

ht flux <u>into</u> fluid $(T_a > T_W)$

Also have radiative transp.

$q^{(r)} = e\sigma T_W^4$ <u>from</u> water <u>to</u> sky

and have radiative transp. <u>from</u> sky <u>to</u> wate

$q^{(a)} = e\sigma T_a^4$ But if clear, dry night, atm is transp. to IR

$\therefore T \ll T_W$ & back rad. may be neglected.

Thus we have energy balance:

$$e\sigma T_W^4 = h(T_a - T_W) = 0.2\,(T_a - T_W)^{5/4}$$

$$\text{or } T_a = T_W + \left(\frac{e\sigma T_W^4}{0.2}\right)^{4/5}$$

For freezing water, IR rad, $e = 0.95$ (nearly 1.)

$\sigma = 0.1712 \times 10^{-8} \ \frac{Btu}{hr\,ft^2\,{}^\circ R^4}$

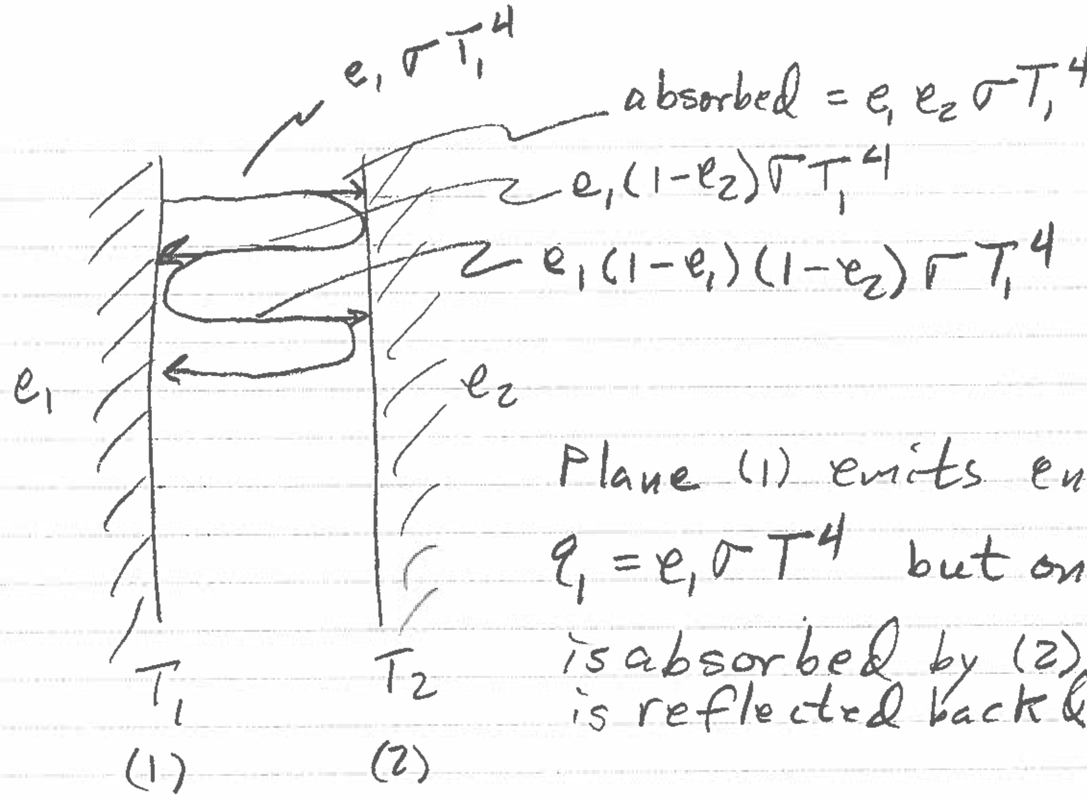
$T_W = 492^\circ R$

$\therefore T_a = T_W + 138^\circ F = \underline{170^\circ F}$

which is surprisingly large. Will not be as great if air is moist (cloudy). Condensation & back radiation both contribute to ht. transf

Still, shows why desert gets cold at night & why temp. fluctuations are moderated on a cloudy day: back radiation.

Look at another problem: ht. transf. between two infinite gray planes (sim. to BS&L 14.5-1)



Plane (1) emits energy flux $q_1 = e_1 \sigma T^4$ but only part is absorbed by (2); some is reflected back & forth

Energy transferred from (1) to (2) is:

$$e_1 e_2 \sigma T_1^4 \sum_{i=0}^{\infty} (1-e_1)^i (1-e_2)^i$$

$$= e_1 e_2 \sigma T_1^4 \frac{1}{1-(1-e_1)(1-e_2)} = \frac{\sigma T_1^4}{\frac{1}{e_1} + \frac{1}{e_2} - 1}$$

and energy from (2) to (1) is:

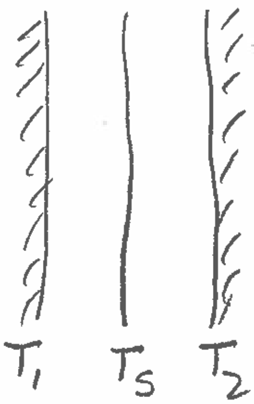
$$\frac{\sigma T_2^4}{\frac{1}{e_1} + \frac{1}{e_2} - 1}$$

Thus the <u>net</u> rate of energy transfer is

$$q_{12} = \frac{\sigma (T_1^4 - T_2^4)}{\left(\frac{1}{e_1} + \frac{1}{e_2} - 1\right)}$$

If emissivities or temp is high, rad ht transf. can be quite significant.

May reduce by putting radiation shield between surfaces, i.e.:



Now $q_{1s} = \dfrac{\sigma (T_1^4 - T_s^4)}{\frac{1}{e_1} + \frac{1}{e_s} - 1}$

and $q_{s2} = \dfrac{\sigma (T_s^4 - T_2^4)}{\frac{1}{e_2} + \frac{1}{e_s} - 1}$

At S.S. $q_{1s} = q_{s2} = q_{12}^{(s)}$

Thus $q_{12}^{(s)} = \dfrac{\sigma (T_1^4 - T_2^4)}{\left(\frac{1}{e_1} + \frac{1}{e_s} - 1\right) + \left(\frac{1}{e_2} + \frac{1}{e_s} - 1\right)}$ ↗ solve for $\underline{T_s}$

or, if $e_1 = e_2 = e$,

$$q_{12}^{(s)} = \frac{1}{2} \frac{\sigma (T_1^4 - T_2^4)}{\left(\frac{1}{e} + \frac{1}{e_s} - 1\right)}$$

Thus we have the improvement:

$$\frac{q_{12}^{(s)}}{q_{12}} = \frac{1}{2} \frac{\left(\frac{2}{e} - 1\right)}{\left(\frac{1}{e} + \frac{1}{e_s} - 1\right)}$$

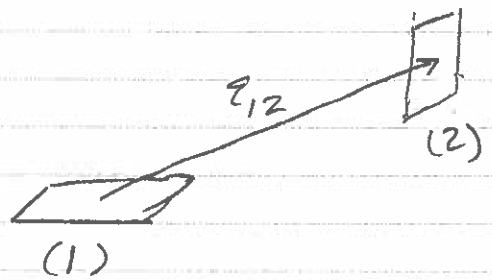
End lec. 27

even if $e_s = e$, gives factor of 2 imp.
∴ multiple shields used for high temp. ins.

Skip in '89 $\Rightarrow$ to p. 203

Thus far, looked at situations w/ no ambiguity as to where radiation goes: all emitted by (1) strikes surface (2) or goes off to ∞

In more general situations, need to consider geometry in calc. ht. transf.



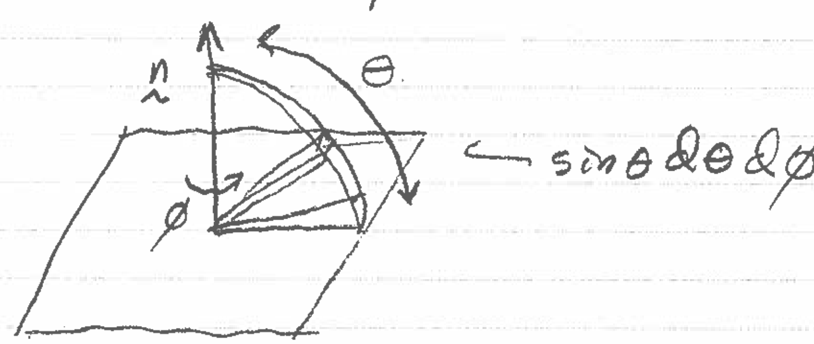
What is q_{12} ?

to solve, must know distrib. w.r.t. angle from unit normal to surface of black body radiation

From geometry & fact that radiation is <u>isotropic</u>, can derive Lambert's law:

$$q_{b\theta}^{(e)} = \frac{q_b^{(e)}}{\pi} \cos\theta = \frac{\sigma T^4}{\pi} \cos\theta$$

where $q_{b\theta}^{(e)}$ is energy flux from unit surface area in direction θ per unit solid angle $\sin\theta\, d\theta\, d\phi$



Integrating $q_{b\theta}^{(e)}$ over hemisphere $0 < \theta < \frac{\pi}{2}, 0 < \phi < \ldots$ gives total flux:

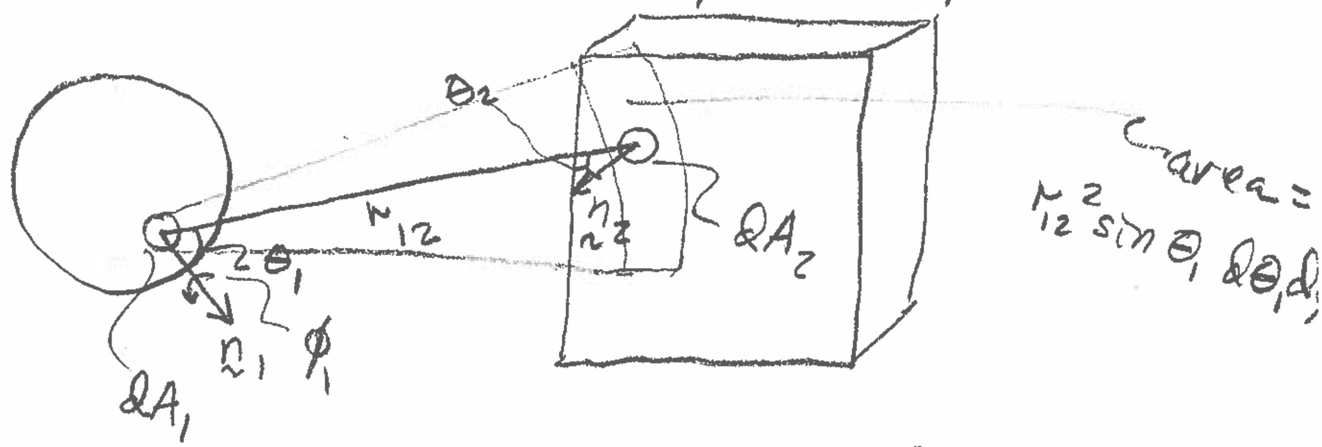
$$q_b^{(e)} = \int_0^{2\pi}\int_0^{\pi/2} q_{b\theta}^{(e)} \sin\theta \, d\theta \, d\phi = \frac{\sigma T^4}{\pi}\int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta d\theta$$

$$= \sigma T^4$$

May now calculate ht. transf. between any two black bodies: Simply compute ht. transf. between dA_1 & dA_2 & integrate over surfaces.

Easier said than done ⇒ rather messy

Consider interaction of sphere & plane:



Energy radiated by dA_1 to solid angle containing dA_2 will be:

$$\left(\frac{\sigma T_1^4}{\pi}\cos\theta_1\right) dA_1 \sin\theta_1 \, d\theta_1 \, d\phi_1$$

Of this energy, only fraction will fall on dA_2.

Fraction given by:

$$\frac{dA_2 \cos\theta_2}{r_{12}^2 \sin\theta_1 \, d\theta_1 \, d\phi_1} \leftarrow \text{projected area}$$

Thus $dQ_{\vec{12}} = \frac{\sigma T_1^4}{\pi} \frac{\cos\theta_1 \cos\theta_2}{r_{12}^2} dA_1 dA_2$

and, from symmetry:

$$dQ_{\vec{21}} = \frac{\sigma T_2^4}{\pi} \frac{\cos\theta_1 \cos\theta_2}{r_{12}^2} dA_1 dA_2$$

Thus <u>net</u> energy transport from (1) to (2) is:

$$dQ_{12} = dQ_{\vec{12}} - dQ_{\vec{21}} = \frac{\sigma}{\pi}\left(T_1^4 - T_2^4\right) \frac{\cos\theta_1 \cos\theta_2}{r_{12}^2} dA_1$$

Obtain <u>total flux</u> by integrating over all areas where surfaces are in view, i.e.

$0 < \theta_1, \theta_2 < \frac{\pi}{2}$

We may thus define the <u>view factors</u>:

$$Q_{12} = \frac{\sigma}{\pi}\left(T_1^4 - T_2^4\right) \iint \frac{\cos\theta_1 \cos\theta_2}{r_{12}^2} dA_1 dA_2$$

$$\equiv A_1 F_{12} \sigma \left(T_1^4 - T_2^4\right) = A_2 F_{21} \sigma \left(T_1^4 - T_2^4\right)$$

where

$$F_{12} = \frac{1}{\pi A_1} \iint \frac{\cos\theta_1 \cos\theta_2}{r_{12}^2} dA_1 dA_2$$

$$F_{21} = \frac{1}{\pi A_2} \iint \frac{\cos\theta_1 \cos\theta_2}{r_{12}^2} dA_1 dA_2$$

Physically, F_{12} is fraction of radiation leaving A_1 that is directly intercepted by A_2

$\therefore F_{12} \leq 1$

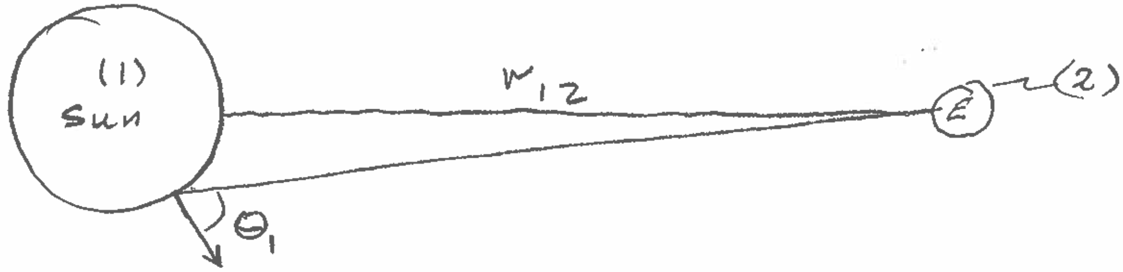
Also, from definition $A_1 F_{12} = A_2 F_{21}$

Very difficult to calculate F_{12} except for simple geometries, however for some useful geometries, graphs of F_{12} have been prepared.

Two ex. in BS&L 14.4-3, 14.4-4

Let's use these view factors to calculate the solar constant:

What is the total radiant energy flux entering earth's atm?



$D_1 = 8.6 \times 10^5$ mi, $r_{12} = 9.29 \times 10^7$ mi

$T_1 = 10{,}400°R$

$\therefore q_{b1} = \sigma T^4 = (0.1712 \times 10^{-8})(10{,}400°R)^4 = 2.0 \times 10^7$ Btu/hr ft^2

$$\frac{dQ_{\vec{12}}}{\cos\theta_2 \, dA_2} = \frac{\sigma T_1^4}{\pi} \int \frac{\cos\theta_1}{r_{12}^2} dA_1$$