Name: ________________________

Instructor: ________________________

Exam 2, Math 20580
March 7, 2024

- The Honor Code *is* in affect for this examination. All work is to be your own.
- Please turn off and stow all cellphones and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 75 minutes.
- Be sure that your name and your instructor's name and your section number are on the front page of your exam.
- There are 12 problems, 8 are multiple choice and 4 are partial credit.
- Be sure that you have all 8 pages of this exam.
- Multiple choice questions should have distinct answers. (If for some reason you think this is not the case, let your instructor know *after the exam* and do one of the following. If you think a multiple choice question has no listed correct answer, leave the line blank. If you are right you will get full credit. If you think a multiple choice question has more than one correct answer, X ONE of them: you will never get credit for a line with two or more X's.)

Please mark you answers with an **X**, not a circle.

1.	(**a**)	(**b**)	(**c**)	(**d**)	(**e**)
2.	(**a**)	(**b**)	(**c**)	(**d**)	(**e**)
			1		
3.	(**a**)	(**b**)	(**c**)	(**d**)	(**e**)
4.	(**a**)	(**b**)	(**c**)	(**d**)	(**e**)
			2		
5.	(**a**)	(**b**)	(**c**)	(**d**)	(**e**)
6.	(**a**)	(**b**)	(**c**)	(**d**)	(**e**)
			3		
7.	(**a**)	(**b**)	(**c**)	(**d**)	(**e**)
8.	(**a**)	(**b**)	(**c**)	(**d**)	(**e**)
			4		

MC Total.	
9.	
10.	
11.	
12.	
Total.	

Multiple choice problems

1. (7 points) Let $\mathcal{P}_1$ denote the vector space of polynomials of degree at most 1, and let $T\colon \mathcal{P}_1 \to \mathcal{P}_1$ denote the linear transformation such that

$$T(1+x) = 1 - 2x \quad \text{and} \quad T(2-3x) = 2 + x.$$

What is the value of $T(3+2x)$?

(a) $4x - 3$ (b) $2x - 4$ (c) $-x + 6$ (d) $-3x + 5$ (e) $3 - 5x$

2. (7 points) Which of the following is NOT ALWAYS a vector space?

(a) The range of a linear transformation.

(b) The null space of a matrix.

(c) The set of solutions in $\mathbb{R}^n$ of a system of linear equations.

(d) The span of a set of vectors in a vector space.

(e) The kernel of a linear transformation.

3. (7 points) Let A and B be 3×3 matrices such that $\det(A) = 5$ and $\det(B) = 2$. What is the value of $\det(5A^{-1}B^2A^T)$, where A^T denotes the transpose of A?

(a) 25 (b) 20 (c) 100 (d) 500 (e) 50

4. (7 points) Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -2 & 2 \end{bmatrix}.$$

What is the determinant of A?

(a) -2 (b) 0 (c) 3 (d) 2 (e) -3

5. (7 points) Let $\mathcal{P}_3$ denote the vector space of polynomials of degree at most 3. Which of the following statements is true?

I. Any linearly independent set of four vectors in $\mathcal{P}_3$ spans $\mathcal{P}_3$.
II. There is a basis of $\mathcal{P}_3$ with three vectors.
III. $\mathcal{P}_3$ can be spanned by five distinct vectors.

(a) I and III only (b) II only (c) I, II and III (d) I only (e) III only

6. (7 points) Let $M_{2,2}$ denote the vector space of 2×2 matrices and let $S\colon M_{2,2} \to M_{2,2}$ be the linear transformation such that $S(A) = A - A^T$ for each 2×2-matrix A, where A^T denotes the transpose of A. What are the rank and nullity of S?

(a) rank $S = 3$ and nullity $S = 1$ (b) rank $S = 3$ and nullity $S = 3$ (c) rank $S = 1$ and nullity $S = 3$

(d) rank $S = 2$ and nullity $S = 2$ (e) rank $S = 1$ and nullity $S = 1$

7. (7 points) Suppose that for some real number s, the system of linear equations

$$\begin{cases} (s+2)x_1 & + & 2x_2 & = 1 \\ (3s+4)x_1 & + (s+3)x_2 & & = 1 \end{cases}$$

has a unique solution for x_1 and x_2. What is the value of x_2 in terms of s, according to Cramer's rule?

(a) $\dfrac{-2s-2}{s^2-s-2}$ (b) $\dfrac{-s-1}{s^2-s-2}$ (c) $\dfrac{2s+2}{s^2-s-2}$ (d) $2s+2$ (e) $\dfrac{s+1}{s^2-s-2}$

8. (7 points) Let $\mathcal{P}_2$ denote the vector space of polynomials of degree at most 2, and let

$$\mathcal{B} = \{1, 1-x, (1-x)^2\}$$

be a basis of $\mathcal{P}_2$. Which polynomial $p(x)$ has $\mathcal{B}$-coordinates

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}?$$

(a) $p(x) = x^2 - 4x$ (b) $p(x) = x^2 + 2x - 8$ (c) $p(x) = x^2 + 2x$ (d) $p(x) = x^2 - 2x + 8$

(e) $p(x) = x^2$

Partial credit problems

9. (11 points) Consider the vector space $\mathcal{P}_2$ of polynomials of degree at most 2, with standard basis $\mathcal{E} = \{1, x, x^2\}$. Define the polynomials

$$p_1(x) = 3 + x + 2x^2, \quad p_2(x) = 2 + 3x + x^2, \quad p_3(x) = 5 - 3x + 4x^2.$$

(a) Write below the $\mathcal{E}$-coordinate vectors $\vec{v}_i = [\,p_i(x)]_{\mathcal{E}}$ for $i = 1, 2, 3$:

$$\vec{v}_1 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}.$$

(b) Find all real numbers c such that $\vec{v}_1 = c\vec{v}_2 + (1-c)\vec{v}_3$.

(c) Use (b) to find all real numbers k such that $p_1(x) = kp_2(x) + (1-k)p_3(x)$. Explain your reasoning.

10. (11 points) Let $\mathcal{P}_2$ denote the vector space of polynomials of degree 2 or less. Consider the two bases of $\mathcal{P}_2$:

$$\mathcal{B} = \{x^2 + 3, x - 4, 1\} \quad \text{and} \quad \mathcal{C} = \{1 - x^2, x^2 + x, x^2\}.$$

(a) Find the change of basis matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ (recall that this is the matrix such that

$$[\,p(x)]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\,p(x)]_{\mathcal{B}}$$

for all vectors $p(x)$ in $\mathcal{P}_2$).

(b) Use your answer to (a) to find the $\mathcal{C}$-coordinates of the polynomial $p(x)$ in $\mathcal{P}_2$ with $\mathcal{B}$-coordinates

$$[\,p(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}.$$

11. (11 points) Consider the matrix

$$A = \begin{bmatrix} 1 & -s & -s \\ 1 & 1 & 0 \\ s & 1 & 1 \end{bmatrix}$$

where s is a real number.

(a) Compute the determinant of A.

(b) For which values of s is the matrix A invertible?

(c) If the matrix A is invertible, what is the entry $(A^{-1})_{13}$ in the first row and third column of A^{-1}.

12. (11 points) Let $\mathcal{P}_2$ denote the vector space of polynomials of degree at most 2, and $T\colon \mathcal{P}_2 \to \mathbb{R}^2$ be the linear transformation given by

$$T(p(x)) = \begin{bmatrix} p(0) \\ p(2) \end{bmatrix}$$

for $p(x)$ in $\mathcal{P}_2$ (you do not have to explain why T is a linear transformation).

(a) Write down the matrix $\underset{\mathcal{E}\leftarrow\mathcal{B}}{[T]}$ of T with respect to the standard bases

$$\mathcal{B} = \{1, x, x^2\} \text{ of } \mathcal{P}_2 \quad \text{and} \quad \mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \text{ of } \mathbb{R}^2.$$

(b) Find a basis for the range of T.

(c) Find a basis for the kernel of T. Make sure to write each element of your basis as a polynomial.