Name: ____________________

Instructor: ____________________

Exam 3, Math 20580
April 18, 2024

- The Honor Code *is* in affect for this examination. All work is to be your own.
- Please turn off and stow all cellphones and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 75 minutes.
- Be sure that your name and your instructor's name and your section number are on the front page of your exam.
- There are 12 problems, 8 are multiple choice and 4 are partial credit.
- Be sure that you have all 8 pages of this exam.
- Multiple choice questions should have distinct answers. (If for some reason you think this is not the case, let your instructor know *after the exam* and do one of the following. If you think a multiple choice question has no listed correct answer, leave the line blank. If you are right you will get full credit. If you think a multiple choice question has more than one correct answer, X ONE of them: you will never get credit for a line with two or more X's.)

01	Laurence Taylor	8:20-9:15	131 DBRT
02	Han Lu	9:25-10:15	140 DBRT
03	Han Lu	10:30-11:20	140 DBRT
04	Henry Chimal-Dzul	11:30-12:20	138 DBRT
05	Michael Gekhtman	12:50-1:40	127 HH
06	Matthew Dyer	2:00-2:50	127 HH

Please mark you answers with an **X**, not a circle.

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
			1		
3.	(a)	(b)	(c)	(d)	(e)
4	(a)	(b)	(c)	(d)	(e)
			2		
5	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
			3		
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)
			4		

MC Total.	
9.	
10	
11.	
12.	
Total.	

Multiple choice problems

1. (6 points) Let $y(t)$ be the solution of the initial value problem

$$\begin{cases} y'(t) = y^2(y-2)(y-4), \\ y(1) = 3. \end{cases}$$

Which of the following is true?

(a) $\lim_{t\to+\infty} y(t) = 4$ and $\lim_{t\to-\infty} y(t) = 0$

(b) $\lim_{t\to+\infty} y(t) = 0$ and $\lim_{t\to-\infty} y(t) = 2$

(c) $\lim_{t\to+\infty} y(t) = 2$ and $\lim_{t\to-\infty} y(t) = 4$

(d) $\lim_{t\to+\infty} y(t) = 4$ and $\lim_{t\to-\infty} y(t) = 2$

(e) $\lim_{t\to+\infty} y(t) = 2$ and $\lim_{t\to-\infty} y(t) = 0$

2. (6 points) The vector $\vec{v} = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$ is a complex eigenvector of a matrix $\begin{bmatrix} 2 & a \\ -4 & 2 \end{bmatrix}$ where a is an unknown real number. What is the value of a? (Hint: First find the corresponding eigenvalue.)

(a) 1 (b) -1 (c) 2 (d) -2 (e) 0

3. (6 points) Which of the following vectors is an eigenvector with eigenvalue -3 of the matrix

$$\begin{bmatrix} 4 & 5 & 3 \\ 1 & -1 & 3 \\ 3 & 0 & -6 \end{bmatrix}?$$

(a) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

4. (6 points) Which of the following statements about the differential equation

$$y' = (y-1)(y+1)$$

are true?

I. The equation is an ordinary differential equation.
II. The equation is separable.
III. The equation is autonomous.

(a) I only (b) I and II only (c) I, II and III (d) I and III only (e) None

5. (6 points) What is the least squares error of the least squares solution of the equation

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}?$$

(a) $\sqrt{2}$ (b) 1 (c) 2 (d) 0 (e) $\sqrt{3}$

6. (6 points) Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$. Which of the following is the matrix R in the QR-decomposition of A?

(a) $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}$ (b) $\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}$ (c) $\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}$ (d) $\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}$

(e) $\begin{bmatrix} 1/\sqrt{2} & \sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}$

7. (6 points) Which of the following is the general solution of the equation $y' + 4y = 0$?

(a) $y = Ce^{4x}$ (b) $y = C(e^{4x} - e^{-4x})$ (c) $y = e^{4x} + C$ (d) $y = Ce^{-4x}$

(e) $y = e^{-4x} + C$

8. (6 points) Find the projection of the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ on the subspace W of $\mathbb{R}^3$ spanned by the vectors $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Partial credit problems

9. (13 points)

(a) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 5 & 0 & -3 \\ -6 & -1 & 3 \\ 6 & 0 & -4 \end{bmatrix}.$$

(b) The matrix

$$B = \begin{bmatrix} -4 & 6 & 2 \\ 0 & 2 & 0 \\ -3 & 3 & 3 \end{bmatrix}$$

has eigenvalues 2 and -3. Find a basis of $\mathbb{R}^3$ consisting of eigenvectors for B.

(c) Diagonalize B. That is, find a diagonal matrix D and an invertible matrix P such that $B = PDP^{-1}$. (You do NOT have to check that $B = PDP^{-1}$.)

10. (13 points) Consider the basis of $\mathbb{R}^3$ consisting of the vectors $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) Apply the Gram-Schmidt process to $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$ to obtain an orthogonal basis $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ of $\mathbb{R}^3$ with $\vec{v}_1 = \vec{x}_1$.

(b) Normalize the orthogonal basis you found in (a) to find an orthonormal basis $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ of $\mathbb{R}^3$ with $\vec{u}_1 = \frac{1}{\sqrt{3}}\vec{x}_1$.

11. (13 points)

(a) Find the solution of the initial value problem

$$\begin{cases} \dfrac{dy}{dx} + 4xy^2 = 0 \\ y(1) = 1. \end{cases}$$

(b) Find the maximal interval on which the solution in (a) is defined.

12. (13 points) Let A and B be constants.

(a) If

$$y(x) = x + A\sin x + B\cos x,$$

calculate $y'(x)$ and $y''(x)$.

(b) Use your answer to (a) to check that

$$y(x) = x + A\sin x + B\cos x$$

is a solution of the differential equation $y'' + y = x$.

(c) Find A and B if $y(x) = x + A\sin x + B\cos x$ is the solution of the initial value problem

$$\begin{cases} y'' + y = x, \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$$