Name: ______________________

Instructor: ______________________

Exam 2, Math 20580
March 7, 2024

- The Honor Code *is* in affect for this examination. All work is to be your own.
- Please turn off and stow all cellphones and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 75 minutes.
- Be sure that your name and your instructor's name and your section number are on the front page of your exam.
- There are 12 problems, 8 are multiple choice and 4 are partial credit.
- Be sure that you have all 8 pages of this exam.
- Multiple choice questions should have distinct answers. (If for some reason you think this is not the case, let your instructor know *after the exam* and do one of the following. If you think a multiple choice question has no listed correct answer, leave the line blank. If you are right you will get full credit. If you think a multiple choice question has more than one correct answer, X ONE of them: you will never get credit for a line with two or more X's.)

Please mark you answers with an **X**, not a circle.

1.	(a)	(b)	(c)	(d)	(e) X
2.	(a)	(b)	(c) X	(d)	(e)
			1		
3.	(a)	(b)	(c)	(d) X	(e)
4.	(a)	(b)	(c)	(d)	(e) X
			2		
5.	(a) X	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c) X	(d)	(e)
			3		
7.	(a) X	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c) X	(d)	(e)
			4		

MC Total.	________
9.	________
10.	________
11.	________
12.	________
Total.	________

Multiple choice problems

1. (7 points) Let $\mathcal{P}_1$ denote the vector space of polynomials of degree at most 1, and let $T\colon \mathcal{P}_1 \to \mathcal{P}_1$ denote the linear transformation such that

$$T(1+x) = 1-2x \quad \text{and} \quad T(2-3x) = 2+x.$$

What is the value of $T(3+2x)$?

(a) $4x-3$ (b) $2x-4$ (c) $-x+6$ (d) $-3x+5$ (e) $3-5x$

If $3+2x = a(1+x) + b(2-3x)$ $(*)$

then $T(3+2x) = a\,T(1+x) + b\,T(2-3x)$

$= a(1-2x) + b(2+x).$

By $(*)$, $a+2b=3$, $a-3b=2$. So $5b=1$, $b=\frac{1}{5}$, $a=3-2b=\frac{13}{5}$,

$T(3+2x) = \frac{13}{5}(1-2x) + \frac{1}{5}(2+x) = 3-5x$

2. (7 points) Which of the following is NOT ALWAYS a vector space?

(a) The range of a linear transformation.

(b) The null space of a matrix.

(c) The set of solutions in $\mathbb{R}^n$ of a system of linear equations.

(d) The span of a set of vectors in a vector space.

(e) The kernel of a linear transformation.

Only the set of solutions of a homogeneous system of linear equations is always a subspace.
For example, the set of solutions of the system (of one equation) $2x=1$ is $x=\frac{1}{2}$, which is not a subspace of $\mathbb{R}$.

3. (7 points) Let A and B be 3×3 matrices such that $\det(A) = 5$ and $\det(B) = 2$. What is the value of $\det(5A^{-1}B^2A^T)$, where A^T denotes the transpose of A?

(a) 25 (b) 20 (c) 100 (d) 500 (e) 50

$$\det(5A^{-1}B^2A^T) = 5^3 \cdot \det(A)^{-1}\det(B)^2\det(A^T)$$
$$= 5^3 \cdot 5^{-1} \cdot 2^2 \cdot 5$$
$$= 500 \quad \text{since } \det(A^T) = \det(A) = 5.$$

4. (7 points) Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -2 & 2 \end{bmatrix}.$$

What is the determinant of A?

(a) -2 (b) 0 (c) 3 (d) 2 (e) -3

$$\begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -2 & 2 \end{vmatrix} \xrightarrow[R3 \to R3 - R2]{R1 \to R1 + 2R2} \begin{vmatrix} 0 & 3 & -3 \\ -1 & 2 & -1 \\ 0 & -4 & 3 \end{vmatrix}.$$

$$= -(-1)\begin{vmatrix} 3 & -3 \\ -4 & 3 \end{vmatrix} = 9 - 12 = -3$$

5. (7 points) Let $\mathcal{P}_3$ denote the vector space of polynomials of degree at most 3. Which of the following statements is true?

I. Any linearly independent set of four vectors in $\mathcal{P}_3$ spans $\mathcal{P}_3$.
II. There is a basis of $\mathcal{P}_3$ with three vectors.
III. $\mathcal{P}_3$ can be spanned by five distinct vectors.

(a) I and III only (b) II only (c) I, II and III (d) I only (e) III only

$\dim \mathcal{P}_3 = 4$, so any L.I. set of four vectors in $\mathcal{P}_3$ is a basis of $\mathcal{P}_3$ and so spans $\mathcal{P}_3$. So I is true
Since $\dim \mathcal{P}_3 = 4$, any basis of $\mathcal{P}_3$ has 4 vectors; II is false.
$\{1, x, x^2, x^3, 1+x\}$ is a set of 5 distinct vectors which span $\mathcal{P}_3$, so III is true.

6. (7 points) Let $M_{2,2}$ denote the vector space of 2×2 matrices and let $S\colon M_{2,2} \to M_{2,2}$ be the linear transformation such that $S(A) = A - A^T$ for each 2×2-matrix A, where A^T denotes the transpose of A. What are the rank and nullity of S?

(a) rank $S = 3$ and nullity $S = 1$ (b) rank $S = 3$ and nullity $S = 3$ (c) rank $S = 1$ and nullity $S = 3$

(d) rank $S = 2$ and nullity $S = 2$ (e) rank $S = 1$ and nullity $S = 1$

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $S(A) = A - A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & c \\ b & d \end{bmatrix}$
$= \begin{bmatrix} 0 & b-c \\ -(b-c) & 0 \end{bmatrix}$. So $\ker S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ such that } b = c \right\}$
$= \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \right\}$ which has basis $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and dimension 3. So $\text{nullity}(S) = 3$, $\text{rank}(S) = \dim(M_{2,2}) - \text{nullity}(S) = 4 - 3 = 1$

Alternatively, note $\text{range}(S) = \left\{ \begin{bmatrix} 0 & b-c \\ -(b-c) & 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 & e \\ e & 0 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ has $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as basis, so $\text{rank}(S) = 1$ (and $\text{nullity}(S) = 4 - 1 = 3$

7. (7 points) Suppose that for some real number s, the system of linear equations

$$\begin{cases} (s+2)x_1 + 2x_2 = 1 \\ (3s+4)x_1 + (s+3)x_2 = 1 \end{cases}$$

has a unique solution for x_1 and x_2. What is the value of x_2 in terms of s, according to Cramer's rule?

(a) $\dfrac{-2s-2}{s^2-s-2}$ (b) $\dfrac{-s-1}{s^2-s-2}$ (c) $\dfrac{2s+2}{s^2-s-2}$ (d) $2s+2$ (e) $\dfrac{s+1}{s^2-s-2}$

$$x_2 = \frac{\begin{vmatrix} s+2 & 1 \\ 3s+4 & 1 \end{vmatrix}}{\begin{vmatrix} s+2 & 2 \\ 3s+4 & s+3 \end{vmatrix}} = \frac{1\cdot(s+2) - 1\cdot(3s+4)}{(s+2)(s+3)-2(3s+4)} = \frac{-2s-2}{s^2-s-2}$$

8. (7 points) Let $\mathcal{P}_2$ denote the vector space of polynomials of degree at most 2, and let

$$\mathcal{B} = \{1, 1-x, (1-x)^2\}$$

be a basis of $\mathcal{P}_2$. Which polynomial $p(x)$ has $\mathcal{B}$-coordinates

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}?$$

(a) $p(x) = x^2 - 4x$ (b) $p(x) = x^2 + 2x - 8$ (c) $p(x) = x^2 + 2x$ (d) $p(x) = x^2 - 2x + 8$

(e) $p(x) = x^2$

$p(x) = 3\cdot 1 - 4(1-x) + 1(x^2-2x+1)$

$= x^2 + 2x$

Partial credit problems

9. (11 points) Consider the vector space $\mathcal{P}_2$ of polynomials of degree at most 2, with standard basis $\mathcal{E} = \{1, x, x^2\}$. Define the polynomials

$$p_1(x) = 3 + x + 2x^2, \quad p_2(x) = 2 + 3x + x^2, \quad p_3(x) = 5 - 3x + 4x^2.$$

(a) Write below the $\mathcal{E}$-coordinate vectors $\vec{v}_i = [p_i(x)]_{\mathcal{E}}$ for $i = 1, 2, 3$:

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 5 \\ -3 \\ 4 \end{bmatrix}.$$

(b) Find all real numbers c such that $\vec{v}_1 = c\vec{v}_2 + (1-c)\vec{v}_3$.

$$\vec{v_1} = c\vec{v_2} + (1-c)\vec{v_3} = c\vec{v_2} + \vec{v_3} - c\vec{v_3}$$

$$\vec{v_1} - \vec{v_3} = c(\vec{v_2} - \vec{v_3})$$

$$\begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = c \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} \quad \text{ie} \begin{cases} -3c = -2 \\ 6c = 4 \\ -3c = -2 \end{cases} \quad \text{so } c = \frac{2}{3}.$$

(c) Use (b) to find all real numbers k such that $p_1(x) = kp_2(x) + (1-k)p_3(x)$. Explain your reasoning.

Taking $\mathcal{E}$-coordinates, $p_1(x) = k\,p_2(x) + (1-k)\,p_3(x)$ is equivalent to $[p_1(x)]_{\mathcal{E}} = k[p_2(x)]_{\mathcal{E}} + (1-k)[p_3(x)]_{\mathcal{E}}$ ie $\vec{v_1} = k\vec{v_2} + (1-k)\vec{v_3}$. So the values of k satisfying the equation are the same as the values of c satisfying the equation in (b) ie $k = \frac{2}{3}$

10. (11 points) Let $\mathcal{P}_2$ denote the vector space of polynomials of degree 2 or less. Consider the two bases of $\mathcal{P}_2$:

$$\mathcal{B} = \{x^2+3, x-4, 1\} \quad \text{and} \quad \mathcal{C} = \{1-x^2, x^2+x, x^2\}.$$

(a) Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ (recall that this is the matrix such that

$$[p(x)]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,[p(x)]_{\mathcal{B}}$$

for all vectors $p(x)$ in $\mathcal{P}_2$).

$$\left[\underset{\mathcal{E}\leftarrow\mathcal{C}}{P} \,\middle|\, \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 4 & -4 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & -5 & 1 \end{array}\right]$$

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 3 & -4 & 1 \\ 0 & 1 & 0 \\ 4 & -5 & 1 \end{bmatrix}$$

(b) Use your answer to (a) to find the $\mathcal{C}$-coordinates of the polynomial $p(x)$ in $\mathcal{P}_2$ with $\mathcal{B}$-coordinates

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}.$$

$$[p(x)]_{\mathcal{C}} = \begin{bmatrix} 3 & -4 & 1 \\ 0 & 1 & 0 \\ 4 & -5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ -1 \\ 17 \end{bmatrix}$$

11. (11 points) Consider the matrix

$$A = \begin{bmatrix} 1 & -s & -s \\ 1 & 1 & 0 \\ s & 1 & 1 \end{bmatrix}$$

where s is a real number.

(a) Compute the determinant of A.

$$\begin{vmatrix} 1 & -s & -s \\ 1 & 1 & 0 \\ s & 1 & 1 \end{vmatrix} \xrightarrow{R1 \to R1 + sR3} \begin{vmatrix} 1+s^2 & 0 & 0 \\ 1 & 1 & 0 \\ s & 1 & 1 \end{vmatrix}$$

$$= (1+s^2) \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \cdot (1+s^2) = 1 + s^2$$

cofactor expansion along R1

(b) For which values of s is the matrix A invertible?

A is invertible $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow 1+s^2 \neq 0$.

so A is invertible for all (real) values of s.

(c) If the matrix A is invertible, what is the entry $(A^{-1})_{13}$ in the first row and third column of A^{-1}.

$$(A^{-1})_{13} = \frac{1}{\det A} \cdot C_{31} = \frac{1}{s^2+1} (+1) \cdot \begin{vmatrix} -s & -s \\ 1 & 0 \end{vmatrix} = \frac{s}{s^2+1}.$$

12. (11 points) Let $\mathcal{P}_2$ denote the vector space of polynomials of degree at most 2, and $T\colon \mathcal{P}_2 \to \mathbb{R}^2$ be the linear transformation given by

$$T(p(x)) = \begin{bmatrix} p(0) \\ p(2) \end{bmatrix}$$

for $p(x)$ in $\mathcal{P}_2$ (you do not have to explain why T is a linear transformation).

(a) Write down the matrix $\underset{\mathcal{E}\leftarrow\mathcal{B}}{[T]}$ of T with respect to the standard bases

$$\mathcal{B} = \{1, x, x^2\} \text{ of } \mathcal{P}_2 \quad \text{and} \quad \mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \text{ of } \mathbb{R}^2.$$

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{[T]} = \left[[T(1)]_{\mathcal{E}} \; [T(x)]_{\mathcal{E}} \; [T(x^2)]_{\mathcal{E}} \right]$$

$$= \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{E}} \begin{bmatrix} 0 \\ 2 \end{bmatrix}_{\mathcal{E}} \begin{bmatrix} 0 \\ 4 \end{bmatrix}_{\mathcal{E}} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix} = A$$

(b) Find a basis for the range of T.

$$A \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \text{ (RREF)}.$$

Columns 1, 2 are pivot columns of A, so corresponding columns $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ are a basis of $\operatorname{col}(A)$. These are the $\mathcal{E}$-coordinate vectors of the basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ of the range of T.

(c) Find a basis for the kernel of T. Make sure to write each element of your basis as a polynomial.

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0} \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 + 2x_3 = 0 \end{cases}$$

The solution is $x_1 = 0$ $x_2 = -2x_3 = -2s$ $x_3 = s$,

so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$. Thus $\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ is a basis of $\operatorname{null}(A)$,

The polynomial $x^2 - 2x$ with $\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ as its $\mathcal{B}$-coordinates is a basis of the kernel of T.