Name: ____________________

Instructor: ____________________

Exam 3, Math 20580
April 18, 2024

- The Honor Code *is* in affect for this examination. All work is to be your own.
- Please turn off and stow all cellphones and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 75 minutes.
- Be sure that your name and your instructor's name and your section number are on the front page of your exam.
- There are 12 problems, 8 are multiple choice and 4 are partial credit.
- Be sure that you have all 8 pages of this exam.
- Multiple choice questions should have distinct answers. (If for some reason you think this is not the case, let your instructor know *after the exam* and do one of the following. If you think a multiple choice question has no listed correct answer, leave the line blank. If you are right you will get full credit. If you think a multiple choice question has more than one correct answer, X ONE of them: you will never get credit for a line with two or more X's.)

01	Laurence Taylor	8:20-9:15	131 DBRT
02	Han Lu	9:25-10:15	140 DBRT
03	Han Lu	10:30-11:20	140 DBRT
04	Henry Chimal-Dzul	11:30-12:20	138 DBRT
05	Michael Gekhtman	12:50-1:40	127 HH
06	Matthew Dyer	2:00-2:50	127 HH

Please mark you answers with an **X**, not a circle.

1.	(a)	(b)	X (c)	(d)	(e)
2.	X (a)	(b)	(c)	(d)	(e)
			1		
3.	(a)	(b)	(c)	(d)	X (e)
4	(a)	(b)	X (c)	(d)	(e)
			2		
5	X (a)	(b)	(c)	(d)	(e)
6.	(a)	X (b)	(c)	(d)	(e)
			3		
7.	(a)	(b)	(c)	X (d)	(e)
8.	(a)	(b)	X (c)	(d)	(e)
			4		

MC Total.	______
9.	______
10	______
11.	______
12.	______
Total.	______

Multiple choice problems

1. (6 points) Let $y(t)$ be the solution of the initial value problem

$$\begin{cases} y'(t) = y^2(y-2)(y-4), \\ y(1) = 3. \end{cases}$$

Which of the following is true?

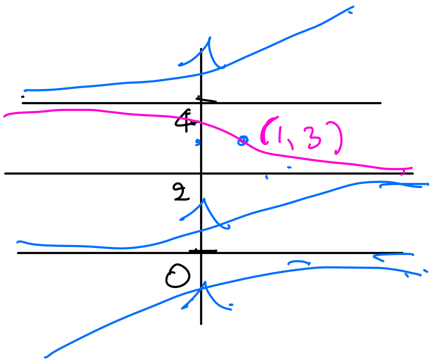
(a) $\lim_{t\to+\infty} y(t) = 4$ and $\lim_{t\to-\infty} y(t) = 0$

(b) $\lim_{t\to+\infty} y(t) = 0$ and $\lim_{t\to-\infty} y(t) = 2$

(c) $\lim_{t\to+\infty} y(t) = 2$ and $\lim_{t\to-\infty} y(t) = 4$

(d) $\lim_{t\to+\infty} y(t) = 4$ and $\lim_{t\to-\infty} y(t) = 2$

(e) $\lim_{t\to+\infty} y(t) = 2$ and $\lim_{t\to-\infty} y(t) = 0$



2. (6 points) The vector $\vec{v} = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$ is a complex eigenvector of a matrix $\begin{bmatrix} 2 & a \\ -4 & 2 \end{bmatrix}$ where a is an unknown real number. What is the value of a? (Hint: First find the corresponding eigenvalue.)

(a) 1 (b) -1 (c) 2 (d) -2 (e) 0

$$\begin{bmatrix} 2 & a \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2i \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

$2 + 2ia = \lambda$

$-4 + 4i = 2i\lambda \rightsquigarrow -2 + 2i = i\lambda$

$\lambda = (-i)(-2+2i) = 2 + 2i$

So $2 + 2ia = 2 + 2i$, $2ia = 2i$, $a = 1$.

3. (6 points) Which of the following vectors is an eigenvector with eigenvalue -3 of the matrix

$$A = \begin{bmatrix} 4 & 5 & 3 \\ 1 & -1 & 3 \\ 3 & 0 & -6 \end{bmatrix}?$$

(a) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

$A\vec{v} = \begin{bmatrix} 8 \\ - \\ - \end{bmatrix}$ $\begin{bmatrix} -6 \\ - \\ - \end{bmatrix}$ $\begin{bmatrix} -1 \\ - \\ - \end{bmatrix}$ $\begin{bmatrix} 12 \\ - \\ - \end{bmatrix}$ $\begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} = -3\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

4. (6 points) Which of the following statements about the differential equation

$$y' = (y-1)(y+1)$$

are true?

I. The equation is an ordinary differential equation.
II. The equation is separable.
III. The equation is autonomous.

(a) I only (b) I and II only (c) I, II and III (d) I and III only (e) None

I ✓

II $\frac{dy}{dx} = (y-1)(y+1)$ $\frac{1}{(y-1)(y+1)}\,dy = dx$. ✓

III $y' = f(y) = (y-1)(y+1)$ ✓

5. (6 points) What is the least squares error of the least squares solution of the equation

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}? \quad A\vec{x} = \vec{b}$$

$\cancel{(a)}$ $\sqrt{2}$ (b) 1 (c) 2 (d) 0 (e) $\sqrt{3}$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(A^TA)\hat{\vec{x}} = A^T\vec{b} = \vec{0} \quad \hat{\vec{x}} = (A^TA)^{-1} \cdot (A^T\vec{b}) = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$A\hat{\vec{x}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{L.S. error} = \|A\hat{\vec{x}} - \vec{b}\| = \left\|\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\| = \sqrt{2}$$

6. (6 points) Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$. Which of the following is the matrix R in the QR-decomposition of A?

(a) $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}$ $\cancel{(b)}$ $\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}$ (c) $\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}$ (d) $\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}$

(e) $\begin{bmatrix} 1/\sqrt{2} & \sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}$

G.S. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ to get $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \dfrac{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$

$\begin{bmatrix} 0 \\ 2 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an orthogonal basis of col(A), $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an orthonormal basis of col(A).

$$Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad R = Q^TA = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}$$

7. (6 points) Which of the following is the general solution of the equation $y' + 4y = 0$?

(a) $y = Ce^{4x}$ (b) $y = C(e^{4x} - e^{-4x})$ (c) $y = e^{4x} + C$ ~~(d)~~ $y = Ce^{-4x}$

(e) $y = e^{-4x} + C$

$$\frac{dy}{dx} + 4y = 0 \qquad \frac{1}{y}\,dy = -4\,dx$$

$$\int \frac{1}{y}\,dy = \int -4\,dx \qquad \ln|y| = -4x + c$$

$$|y| = e^{-4x+c} = e^{c} e^{-4x}$$

$$y = \underbrace{\pm e^{c}}_{C} e^{-4x} = Ce^{-4x}$$

8. (6 points) Find the projection of the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ on the subspace W of $\mathbb{R}^3$ spanned by the vectors $\vec{a} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \vec{b}$. Note $\vec{a} \cdot \vec{b} = 0$

(a) $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$ ~~(c)~~ $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$$\frac{\vec{v} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a} + \frac{v \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b} = \frac{-4}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Partial credit problems

9. (13 points)
(a) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 5 & 0 & -3 \\ -6 & -1 & 3 \\ 6 & 0 & -4 \end{bmatrix}.$$

$$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & 0 & -3 \\ -6 & -1-\lambda & 3 \\ 6 & 0 & -4-\lambda \end{vmatrix} = (-1-\lambda)\begin{vmatrix} 5-\lambda & -3 \\ 6 & -4-\lambda \end{vmatrix}$$

$$= -(\lambda+1)\left[(5-\lambda)(-4-\lambda) - (-3)6\right]$$

$$= -(\lambda+1)(\lambda^2-\lambda-2) = -(\lambda+1)(\lambda+1)(\lambda-2)$$

Roots: $-1, -1, 2$ are the eigenvalues.

(b) The matrix

$$B = \begin{bmatrix} -4 & 6 & 2 \\ 0 & 2 & 0 \\ -3 & 3 & 3 \end{bmatrix}$$

has eigenvalues 2 and -3. Find a basis of $\mathbb{R}^3$ consisting of eigenvectors for B.

$$B-2I = \begin{bmatrix} -6 & 6 & 2 \\ 0 & 0 & 0 \\ -3 & 3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ RREF.}$$

$x_1 - x_2 - \frac{1}{3}x_3 = 0.$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s+\frac{1}{3}t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}.$ Basis of 2-eigenspace: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

$$B+3I = \begin{bmatrix} -1 & 6 & 2 \\ 0 & 5 & 0 \\ -3 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 2x_3 = 0$ $x_2 = 0.$ $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ is basis of -3-eigenspace.

Basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ for $\mathbb{R}^3$ consists of eigenvectors for B.

(c) Diagonalize B. That is, find a diagonal matrix D and an invertible matrix P such that $B = PDP^{-1}$. (You do NOT have to check that $B = PDP^{-1}$.)

$$P = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

10. (13 points) Consider the basis of $\mathbb{R}^3$ consisting of the vectors $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) Apply the Gram-Schmidt process to $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$ to obtain an orthogonal basis $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ of $\mathbb{R}^3$ with $\vec{v}_1 = \vec{x}_1$.

$$\vec{v}_1 = \vec{x}_1, \quad \vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{4}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{0}{8} \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

(b) Normalize the orthogonal basis you found in (a) to find an orthonormal basis $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ of $\mathbb{R}^3$ with $\vec{u}_1 = \frac{1}{\sqrt{3}} \vec{x}_1$.

$$\vec{u}_i = \frac{1}{\|v_i\|} \vec{v}_i. \quad \|\vec{v}_1\| = \sqrt{3}, \ \|\vec{v}_2\| = \sqrt{8} = 2\sqrt{2},$$

$$\|\vec{v}_3\| = \frac{1}{3}\sqrt{6}. \quad \vec{u}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

11. (13 points)

(a) Find the solution of the initial value problem

$$\begin{cases} \dfrac{dy}{dx} + 4xy^2 = 0 \\ y(1) = 1. \end{cases}$$

$\frac{dy}{dx} + 4xy^2 = 0 \qquad 4x\,dx = -y^{-2}dy$

$\int 4x\,dx = \int -y^{-2}dy \qquad y^{-1} = 2x^2 + C.$

When $y=1$, $x=1$, so $1^{-1} = 2\cdot 1^2 + C$, $C=-3$

$y^{-1} = 2x^2 - 3$ and $y(x) = \frac{1}{2x^2-3}$.

(b) Find the maximal interval on which the solution in (a) is defined.

$y(x)$ is undefined if $2x^2-3=0$ ie $x = \pm\sqrt{\frac{3}{2}}$,
The maximal interval can't contain either of these points but must contain the initial x-value $x=1$, so it is

$-\sqrt{\frac{3}{2}} < x < \sqrt{\frac{3}{2}}$ i.e. $\left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$

12. (13 points) Let A and B be constants.

(a) If
$$y(x) = x + A\sin x + B\cos x,$$
calculate $y'(x)$ and $y''(x)$.

$y'(x) = 1 + A\cos x - B\sin x$

$y''(x) = -A\sin x - B\cos x.$

(b) Use your answer to (a) to check that
$$y(x) = x + A\sin x + B\cos x$$
is a solution of the differential equation $y'' + y = x$.

$y'' + y = (-A\sin x - B\cos x) + (x + A\sin x + B\cos x)$

$= x$

(c) Find A and B if $y(x) = x + A\sin x + B\cos x$ is the solution of the initial value problem
$$\begin{cases} y'' + y = x, \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$$

$1 = y(0) = 0 + A\sin 0 + B\cos 0 \rightsquigarrow B = 1$

$0 = y'(0) = 1 + A\cos 0 - B\sin 0 \rightsquigarrow A = -1,$