Part I: Multiple choice questions (7 points each)

1. If the matrices A, B are such that

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \qquad BA = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix},$$

then what is the matrix B?

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} -7 & 5 \\ -11 & 8 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ (e) It can't be determined.

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$B = BA \cdot A^{-1} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 5 \\ -11 & 8 \end{bmatrix}$$

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 5 \end{bmatrix}.$$

Which of the following vectors are in the null space of A?

✓ (I) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (II) $\begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$ X (III) $\begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix}$ ✓ (IV) $\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$ X

(a) (IV) only

(b) (I) and (III) only

(c) (I) and (II) only

(d) (I) and (IV) only

(e) (II), (III) and (IV) only

$$A \cdot \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5-10+5 \\ 15-20+5 \end{bmatrix} = \vec{0} \quad ✓$$

$$A \cdot \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1+2-10 \end{bmatrix} \neq \vec{0} \quad X$$

$\vec{v} = 2\vec{u}$ (b), (d)

$\vec{0}$ vector (a), (e)

3. Consider the following vectors.

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Which of the following sets of vectors is linearly independent?

(a) $\{\cancel{\vec{u}, \vec{w}}\}$

(b) $\{\cancel{\vec{u}, \vec{v}, \vec{x}}\}$

(c) $\{\vec{v}, \vec{x}\}$ ✓

(d) $\{\cancel{\vec{u}, \vec{v}}\}$

(e) $\{\vec{u}, \vec{v}, \cancel{\vec{w}, \vec{x}}\}$

4. Suppose that the null space of a matrix A has dimension 3. Which of the following *must* be true about A?

(a) $A\vec{x} = \vec{b}$ always has a solution for any $\vec{b}$.

(b) $A\vec{x} = \vec{b}$ never has a solution for any $\vec{b}$.

(c) If $A\vec{x} = \vec{b}$ has a solution, it will be unique.

(d) If $A\vec{x} = \vec{b}$ has a solution, it will not be unique.

(e) For some $\vec{b}$, $A\vec{x} = \vec{b}$ has exactly 3 solutions.

3 free variables so if $A\vec{x} = \vec{b}$ consistent then infinitely many solutions!

5. What can be said about the following system of linear equations?

$$\begin{cases} 4x_1 + 6x_3 = 0 \\ 11x_2 - 15x_3 = 0 \end{cases}$$

(a) The solution set is a subspace of $\mathbb{R}^3$ (b) The system is inconsistent

(c) There are only finitely many solutions (d) Every solution is in $\mathbb{R}^2$

(e) none of the above

6. Find the values of k for which the columns of the matrix

$$A = \begin{bmatrix} 1 & 2 & k \\ 2 & k^2 & 4 \end{bmatrix}$$

do not span $\mathbb{R}^2$.

(a) $k = 2$ or $k = -2$ (b) $k = 0$ (c) $k = -1$ (d) $k = 2$

(e) none of the above

$$\begin{bmatrix} 1 & 2 & k \\ 2 & k^2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & k \\ 0 & k^2-4 & 4-2k \end{bmatrix}$$

must be 0

$k^2-4=0$ and $4-2k=0$

So $\underline{k=2}$

7. Consider a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ of $\mathbb{R}^3$, and a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ with the property that

$$T(\vec{b}_1) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad T(\vec{b}_2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad T(\vec{b}_3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

If $\vec{u}$ has coordinate vector $[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ relative to $\mathcal{B}$, then $T(\vec{u})$ is equal to

(a) $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ (c) $2\vec{b}_1 + 4\vec{b}_2 + \vec{b}_3$ (d) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (e) $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$

$$\vec{u} = 2\vec{b}_1 + 4\vec{b}_2 + \vec{b}_3$$

$$T(\vec{u}) = 2\,T(\vec{b}_1) + 4\,T(\vec{b}_2) + T(\vec{b}_3)$$

$$= 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

8. Let A be a 4×8 matrix of rank 2. Which of the following is the dimension of the null space of A?

(a) 0 (b) 2 (c) 4 (d) 6 (e) 8

$$4\Big\{\overbrace{\begin{bmatrix} \quad\quad\quad\quad \end{bmatrix}}^{8}$$

$$8 = \underset{\substack{\parallel \\ 2}}{\text{rank } A} + \dim(\text{null } A)$$

so $\dim(\text{null } A) = 6$

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the linear system

$$\begin{cases} x_1 - x_2 + 3x_4 + x_5 = 2 \\ x_1 + x_2 + 2x_3 + x_4 - x_5 = 4 \\ x_2 + 2x_4 + 3x_5 = 0 \end{cases}$$

(a) Write down the coefficient matrix and the augmented matrix of the system.

$$\begin{bmatrix} 1 & -1 & 0 & 3 & 1 \\ 1 & 1 & 2 & 1 & -1 \\ 0 & 1 & 0 & 2 & 3 \end{bmatrix} \qquad \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 & -1 & 4 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{array}\right]$$

coefficient matrix augmented matrix

$R_2 \to R_2 - R_1$

(b) Describe the solution set of the system in parametric vector form.

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 2 & 2 & -2 & -2 & 2 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{2} R_2} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 3 & 4 & -1 \end{array}\right]$$

free

$x_4 = s$
$x_5 = t$ free

$-x_3 + 3x_4 + 4x_5 = -1$

$\Rightarrow \boxed{x_3 = 3s + 4t + 1}$

$x_2 + x_3 - x_4 - x_5 = 1 \Rightarrow \boxed{x_2 = -2s - 3t}$

$x_1 - x_2 + 3x_4 + x_5 = 2 \Rightarrow \boxed{x_1 = -5s - 4t + 2}$

$$\vec{x} = \begin{bmatrix} -5s-4t+2 \\ -2s-3t \\ 3s+4t+1 \\ s \\ t \end{bmatrix} = s \cdot \begin{bmatrix} -5 \\ -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ -3 \\ 4 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

10. Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 5 & -4 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & 5 & -4 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 5 & -4 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 5 & -4 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_1 \to R_1 + 4R_3]{R_2 \to R_2 + 2R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 0 & -4 & 1 & 4 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - 5R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -6 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & -6 \\ -1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

11. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 3 & 1 \\ -1 & -1 & -1 & 1 \\ 1 & 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \to$$

(a) Find the rank of A, find a basis $\mathcal{B}$ for $\mathrm{Col}(A)$, and find a basis $\mathcal{R}$ for $\mathrm{Row}(A)$.

$$\to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank $A = 3$

$$\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}}_{\vec{a}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}}_{\vec{a}_2}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}}_{\vec{a}_4} \right\} \qquad \mathcal{R} = \{ [1\ 0\ 2\ 0], [0\ 1\ -1\ 1], [0\ 0\ 0\ 2] \}$$

(b) For the basis $\mathcal{B}$ found in part (a), determine the coordinate vector $[\vec{v}]_{\mathcal{B}}$, where $\vec{v} = \vec{a}_2 + 2\vec{a}_3 - \vec{a}_4$ and $\vec{a}_i$ denotes the i-th column of the matrix A.

since $\left[\vec{a}_1 \ \vec{a}_2 \,\middle|\, \vec{a}_3 \right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$

we get $\vec{a}_3 = 2\vec{a}_1 - \vec{a}_2$

Therefore $\vec{v} = \vec{a}_2 + 2(2\vec{a}_1 - \vec{a}_2) - \vec{a}_4$

$= 4\vec{a}_1 - \vec{a}_2 - \vec{a}_4$

and $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$

Alternatively:

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -4 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 2 & 1 & 1 & 6 \\ -1 & -1 & 1 & -4 \\ 1 & 0 & 2 & 2 \end{array} \right] \to \cdots$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \text{So } [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$$

12. (a) Find a 3×3 matrix whose null space contains all the vectors of the form $\begin{bmatrix} r \\ r \\ r \end{bmatrix}$ $\Rightarrow \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ is contained in null space

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(or any matrix $A = [\vec{a_1} \; \vec{a_2} \; \vec{a_3}]$ with $\vec{a_1} + \vec{a_2} + \vec{a_3} = \vec{0}$)

(b) Find a 3×3 matrix whose column space contains all the vectors of the form $\begin{bmatrix} s+t \\ s-t \\ 2t \end{bmatrix}$.

$$= \left\{ s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\} = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right\}$$

is contained in the column space

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

(c) Find a 3×3 matrix whose null space consists of all vectors of the form $\begin{bmatrix} r \\ r \\ r \end{bmatrix}$, and whose column space consists of all vectors of the form $\begin{bmatrix} s+t \\ s-t \\ 2t \end{bmatrix}$.

As in part (a), we need $\vec{a_3} = -\vec{a_1} - \vec{a_2}$

Can take $\vec{a_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\vec{a_2} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ as in part (b)

and obtain

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 0 \\ 0 & 2 & -2 \end{bmatrix}$$