Math 20580 Name:____________________
Midterm 2 Instructor:____________________
March 5, 2020 Section:____________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. a b c d e
2. a b c d e
3. a b c d e
4. a b c d e
5. a b c d e
6. a b c d e
7. a b c d e
8. a b c d e

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. Assume that A and B are two 4×4 matrices with determinants $\det A = 2$, $\det B = 3$. Find the determinant $\det(A^T B A^{-1} B)$.

   (a) 0 (b) 9/4 (c) 36 (d) 9 (e) cannot be determined.

2. Consider the four functions $f_1 = (\sin t)^2$, $f_2 = (\cos t)^2$, $f_3 = 1$, $f_4 = \cos 2t$. They generate a subspace $H = \text{Span}\{f_1, f_2, f_3, f_4\}$ in the vector space $C[0,1]$ of continuous functions on the interval $[0,1]$. Which among the following sets is a basis for H?

   Hint: You may use the trig identity
   $\cos 2t = (\cos t)^2 - (\sin t)^2 = 2(\cos t)^2 - 1 = 1 - 2(\sin t)^2$.

   (a) $\{f_1, f_2\}$ (b) $\{f_1, f_2, f_3\}$ (c) $\{f_1, f_2, f_3, f_4\}$ (d) $\{f_1\}$
   (e) none of the above.

3. Which among the following subsets of $\mathbb{R}^3$ is a subspace?

1. $\left\{ \begin{bmatrix} t \\ s \\ \sin t \end{bmatrix} \middle| t, s \in \mathbb{R} \right\}$

2. $\left\{ \begin{bmatrix} t \\ 2t \\ 1 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$

3. $\left\{ \begin{bmatrix} t \\ s \\ t+s \end{bmatrix} \middle| t \in \mathbb{R}, s \geq 0 \right\}$

4. $\left\{ \begin{bmatrix} t \\ t+s \\ s \end{bmatrix} \middle| t, s \in \mathbb{R} \right\}$

(a) 3 and 4 only (b) 4 only (c) 1, 3 and 4 only

(d) all of them (e) none of them.

4. Let S be the parallogram determined by the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Find the area of S.

(a) 0 (b) 1 (c) -1 (d) 42 (e) none of the above.

5. Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by $T : p(t) \mapsto t\,p'(t) - p(t)$. Which of the following polynomials is in the null space (kernel) of T?

(I) t^2 (II) $2t$ (III) $1 - t^2$ (IV) $-t$

(a) I and II only (b) IV only (c) III only (d) I and III only (e) II and IV only

6. Let H be the subspace of $\mathbb{P}_3$ consisting of all polynomials $p(t)$ of degree at most 3 such that $p(-1) = 0$. Which of the following is a basis of H?

(a) $\{1, t, t^2, t^3\}$ (b) $\{t - 1, t^2 + 1, t^3 - 1\}$ (c) $\{t + 1, t^2 - 1, t^3 + 1\}$

(d) $\{t + 1, t^2 - 1, t^2 + t^3, t^3 + 1\}$ (e) $\{t + 1, t^3 + 1\}$

7. Let $T\colon \mathbb{R}^{12} \to \mathbb{R}^8$ be a linear transformation of $\mathbb{R}^{12}$ *onto* $\mathbb{R}^8$. What is the dimension of the null space (kernel) of T?

(a) 3 (b) 4 (c) 6 (d) 8 (e) 11.

8. The vector $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix $A = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 2 & 0 \\ 1 & -4 & 10 \end{bmatrix}$. What is the corresponding eigenvalue?

(a) 2 (b) 6 (c) 4 (d) 8 (e) 3

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the matrix

$$A = \begin{bmatrix} s & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & s \end{bmatrix}$$

with s a parameter.

(a) Calculate the determinant of A.

(b) For which values of the parameter s is the matrix A invertible?

(c) When A is invertible, find the entry in row 1, column 3 of the inverse matrix A^{-1} (the formula will depend on the parameter s).

10. Consider the two ordered bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^3$ given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

(a) Find the change of coordinate matrix $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$ from $\mathcal{C}$ to $\mathcal{B}$ (recall that $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$ is the matrix such that $[\vec{x}]_{\mathcal{B}} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P} \cdot [\vec{x}]_{\mathcal{C}}$ for all vectors $\vec{x}$ in $\mathbb{R}^3$).

(b) If $\vec{v}$ is a vector in $\mathbb{R}^3$ with $[\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, determine $[\vec{v}]_{\mathcal{B}}$ and $\vec{v}$.

11. Let A be the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) Find all the eigenvalues of A.

(b) For each eigenvalue of A, determine a basis of the corresponding eigenspace.

12. Consider the vector space $\mathbb{P}_2$ of polynomials of degree at most 2 in the variable t. Let $\mathcal{E} = \{1, t, t^2\}$ denote the standard basis of $\mathbb{P}_2$. Define the following polynomials in $\mathbb{P}_2$:

$$p_0(t) = 1+t+t^2, \qquad p_1(t) = 1+2t+3t^2, \qquad p_2 = 1+4t+9t^2, \qquad p_3 = 1+8t+17t^2.$$

(a) Write below the coordinate vector $\vec{v}_i = [p_i(t)]_{\mathcal{E}}$ in $\mathbb{R}^3$ for each $i = 0, 1, 2, 3$.

$$\vec{v}_0 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}.$$

(b) Find scalars a_0, a_1, a_2 such that $\vec{v}_3 = a_0\vec{v}_0 + a_1\vec{v}_1 + a_2\vec{v}_2$.

(c) Find scalars b_0, b_1, b_2 such that $p_3 = b_0p_0 + b_1p_1 + b_2p_2$. Explain your reasoning.