Math 20580 Name:____________________

Midterm 2 Instructor:____________________

March 5, 2020 Section:____________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. a b c ~~d~~ e
2. ~~a~~ b c d e
3. a ~~b~~ c d e
4. a ~~b~~ c d e
5. a b c d ~~e~~
6. a b ~~c~~ d e
7. a ~~b~~ c d e
8. a b c d ~~e~~

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. Assume that A and B are two 4×4 matrices with determinants $\det A = 2$, $\det B = 3$. Find the determinant $\det(A^T B A^{-1} B)$.

(a) 0 (b) 9/4 (c) 36 (d) 9 (e) cannot be determined.

$\det(A^T) \cdot \det(B) \cdot \det(A^{-1}) \cdot \det B =$

$2 \cdot 3 \cdot \frac{1}{2} \cdot 3 = 9$

2. Consider the four functions $f_1 = (\sin t)^2$, $f_2 = (\cos t)^2$, $f_3 = 1$, $f_4 = \cos 2t$. They generate a subspace $H = \text{Span}\{f_1, f_2, f_3, f_4\}$ in the vector space $C[0,1]$ of continuous functions on the interval $[0,1]$. Which among the following sets is a basis for H?

Hint: You may use the trig identity
$\cos 2t = (\cos t)^2 - (\sin t)^2 = 2(\cos t)^2 - 1 = 1 - 2(\sin t)^2$.

(a) $\{f_1, f_2\}$ (b) $\{f_1, f_2, f_3\}$ (c) $\{f_1, f_2, f_3, f_4\}$ (d) $\{f_1\}$
(e) none of the above.

$f_3 = f_1 + f_2$

$f_4 = f_2 - f_1$

$\Rightarrow f_1, f_2$ $\boxed{\text{Span}}$

f_1, f_2 not scalar multiples of each other so $\{f_1, f_2\}$ $\boxed{\text{independent}}$

3. Which among the following subsets of $\mathbb{R}^3$ is a subspace?

1. $\left\{ \begin{bmatrix} t \\ s \\ \sin t \end{bmatrix} \middle| t, s \in \mathbb{R} \right\}$ ✗

2. $\left\{ \begin{bmatrix} t \\ 2t \\ 1 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$ ✗

3. $\left\{ \begin{bmatrix} t \\ s \\ t+s \end{bmatrix} \middle| t \in \mathbb{R}, s \geq 0 \right\}$ ✗

4. $\left\{ \begin{bmatrix} t \\ t+s \\ s \end{bmatrix} \middle| t, s \in \mathbb{R} \right\}$ $= \left\{ t \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \middle| t, s \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ ✓

(a) 3 and 4 only (b) 4 only (c) 1, 3 and 4 only

(d) all of them (e) none of them.

4. Let S be the parallogram determined by the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Find the area of S.

(a) 0 (b) 1 (c) -1 (d) 42 (e) none of the above.

$\left| \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \right| = |-1| = 1$

5. Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by $T : p(t) \mapsto t\,p'(t) - p(t)$. Which of the following polynomials is in the null space (kernel) of T?

(I) t^2 (II) $2t$ (III) $1 - t^2$ (IV) $-t$

(a) I and II only (b) IV only (c) III only (d) I and III only (e) II and IV only

$t^2 \to t \cdot 2t - t^2 = t^2 \neq 0$

$2t \to t \cdot 2 - 2t = 0$ ✓

$1-t^2 \to t(-2t) - (1-t^2) = -t^2 - 1 \neq 0$

$-t \to t \cdot (-1) + t = 0$ ✓

6. Let H be the subspace of $\mathbb{P}_3$ consisting of all polynomials $p(t)$ of degree at most 3 such that $p(-1) = 0$. Which of the following is a basis of H?

(a) $\{1, t, t^2, t^3\}$ (b) $\{t - 1, t^2 + 1, t^3 - 1\}$ (c) $\{t + 1, t^2 - 1, t^3 + 1\}$

(d) $\{t + 1, t^2 - 1, t^2 + t^3, t^3 + 1\}$ (e) $\{t + 1, t^3 + 1\}$

$p(t) = a + bt + ct^2 + dt^3$

$p(-1) = 0 \iff a - b + c - d = 0 \iff a = b - c + d$

So $p(t) = (b - c + d) + bt + ct^2 + dt^3$

$= b(\underline{1+t}) + c(\underline{t^2-1}) + d(\underline{t^3+1})$

Span and independent ✓

7. Let $T: \mathbb{R}^{12} \to \mathbb{R}^8$ be a linear transformation of $\mathbb{R}^{12}$ *onto* $\mathbb{R}^8$. What is the dimension of the null space (kernel) of T?

(a) 3 (b) 4 (c) 6 (d) 8 (e) 11.

$$12 = 8 + \underbrace{\dim \mathrm{Nul}(T)}_{4}$$

8. The vector $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix $A = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 2 & 0 \\ 1 & -4 & 10 \end{bmatrix}$. What is the corresponding eigenvalue?

(a) 2 (b) 6 (c) 4 (d) 8 (e) 3

$$\begin{bmatrix} 4 & -2 & 3 \\ 2 & 2 & 0 \\ 1 & -4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ \vdots \\ \end{bmatrix} = \lambda \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

So $\lambda = 3$

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the matrix

$$A = \begin{bmatrix} s & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & s \end{bmatrix}$$

with s a parameter.

(a) Calculate the determinant of A.

Expand along row 1:

$$\det A = s \cdot \det \begin{bmatrix} 1 & 1 \\ 1 & s \end{bmatrix} - 0 + 1 \cdot \underbrace{\det \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}}_{=0}$$

$$\boxed{= s(s-1)}$$

(b) For which values of the parameter s is the matrix A invertible?

$\det A \neq 0$, so $\boxed{\begin{cases} s \neq 0 \\ s \neq 1 \end{cases}}$

(c) When A is invertible, find the entry in row 1, column 3 of the inverse matrix A^{-1} (the formula will depend on the parameter s).

$(1,3)$-entry of A^{-1} is $\dfrac{C_{3,1}}{\det A}$

$C_{3,1} = (-1)^{3+1} \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = -1$

$\Rightarrow \dfrac{-1}{s(s-1)}$

10. Consider the two ordered bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^3$ given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

(a) Find the change of coordinate matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ from $\mathcal{C}$ to $\mathcal{B}$ (recall that $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ is the matrix such that $[\vec{x}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} \cdot [\vec{x}]_{\mathcal{C}}$ for all vectors $\vec{x}$ in $\mathbb{R}^3$).

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 & -1 \end{array}\right] \xleftrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$$

(b) If $\vec{v}$ is a vector in $\mathbb{R}^3$ with $[\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, determine $[\vec{v}]_{\mathcal{B}}$ and $\vec{v}$.

$$[\vec{v}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} [\vec{v}]_{\mathcal{C}}$$

$$= \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} //$$

$$\vec{v} = \vec{b}_2 + \vec{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} //$$

Alternatively, $\vec{v} = \vec{c}_1 + \vec{c}_2 + \vec{c}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} //$

11. Let A be the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) Find all the eigenvalues of A.

$\lambda = 0$ (multiplicity 2)

$\lambda = 2$

$$0 = \det(A - \lambda I_3) = \det \begin{bmatrix} -\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}$$

$$= (-\lambda) \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (-\lambda)\left((1-\lambda)^2 - 1\right) = -\lambda \cdot \lambda \cdot (\lambda - 2)$$

(b) For each eigenvalue of A, determine a basis of the corresponding eigenspace.

$$E_0 = \mathrm{Nul} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = s$

$x_2 = -t$

$x_3 = t$

$$\vec{x} = \begin{bmatrix} s \\ -t \\ t \end{bmatrix} = s \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$\boxed{\text{So Basis for } E_0 \text{ is } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}}$

$$E_2 = \mathrm{Nul} \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \ldots = \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$\boxed{\text{basis for } E_2 \text{ is } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}}$

12. Consider the vector space $\mathbb{P}_2$ of polynomials of degree at most 2 in the variable t. Let $\mathcal{E} = \{1, t, t^2\}$ denote the standard basis of $\mathbb{P}_2$. Define the following polynomials in $\mathbb{P}_2$:

$$p_0(t) = 1 + t + t^2, \qquad p_1(t) = 1 + 2t + 3t^2, \qquad p_2 = 1 + 4t + 9t^2, \qquad p_3 = 1 + 8t + 17t^3.$$

(a) Write below the coordinate vector $\vec{v}_i = [p_i(t)]_{\mathcal{E}}$ in $\mathbb{R}^3$ for each $i = 0, 1, 2, 3$.

$$\vec{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 8 \\ 17 \end{bmatrix}.$$

(b) Find scalars a_0, a_1, a_2 such that $\vec{v}_3 = a_0\vec{v}_0 + a_1\vec{v}_1 + a_2\vec{v}_2$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 17 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 2 & 8 & 16 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 2 \end{array}\right] \begin{array}{l} \to a_0 + a_1 + a_2 = 1 \to \boxed{a_0 = -4} \\ \to a_1 + 3a_2 = 7 \to \boxed{a_1 = 4} \\ \to \boxed{a_2 = 1} \end{array}$$

(c) Find scalars b_0, b_1, b_2 such that $p_3 = b_0p_0 + b_1p_1 + b_2p_2$. Explain your reasoning.

Since $\vec{v}_i = [p_i]_{\mathcal{E}}$ same weights work

$$\boxed{b_0 = -4, \quad b_1 = 4, \quad b_2 = 1}$$