Math 20580
Midterm 3
April 20, 2023

Name:________________________
Instructor:________________________
Section:________________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

____

1. a b c d e
2. a b c d e
3. a b c d e
4. a b c d e
5. a b c d e
6. a b c d e
7. a b c d e
8. a b c d e

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. Which of the following is an eigenvalue of $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ corresponding to the eigenvector $\vec{x} = \begin{bmatrix} i \\ 1 \end{bmatrix}$?

   (a) $\lambda = 1 + 3i$ (b) $\lambda = i$ (c) $\lambda = 1 + i$ (d) $\lambda = 1$ (e) $\lambda = -1$

2. Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + y - z = 0 \right\}$. What is the dimension of the orthogonal complement $W^\perp$ of W?

   (a) 3 (b) 0 (c) 2 (d) 1 (e) None

3. Consider the orthogonal vectors $\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The distance from the vector $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ to $W = \text{Span}\{\vec{w}_1, \vec{w}_2\}$ is:

(a) $\sqrt{6}$ (b) 1 (c) $\sqrt{2}$ (d) 0 (e) $\sqrt{3}$

4. The matrix $A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$ factors as $A = QR$, where $Q = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ -1/\sqrt{2} & 2/3 \\ 0 & 1/3 \end{bmatrix}$ and R is:

(a) $\begin{bmatrix} 1/\sqrt{2} & 1/3 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} \sqrt{3} & 1 \\ 0 & -\sqrt{2} \end{bmatrix}$ (e) $\begin{bmatrix} 2/3 & 1/3 \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$

5. Let $y(x)$ be the unique solution of the initial value problem

$$\sqrt{4-x^2}\,y'' + \frac{x}{x^2+1}\,y' + 5y = \frac{1}{x^2-5x+4}, \quad y(0)=1,\ y'(0)=7.$$

What is the largest interval where $y(x)$ is defined?

(a) $x \geq 0$ (b) $-2 \leq x \leq 2$ (c) $-2 < x < 1$ (d) $-2 < x < 2$ (e) $x < 2$

6. Find all stable critical values (also known as stable equilibrium solutions) for the autonomous system

$$\frac{dy}{dx} = y^2(y-3)(y+2).$$

(a) $y=3,\ y=0,\ y=-2$ (b) $y=-2$ (c) $y=3,\ y=0$

(d) $y=0$ (e) $y=3$

7. Solve the initial value problem

$$\frac{dy}{dx} - \frac{3}{x}y = x^6, \qquad y(1) = \frac{5}{4}.$$

(a) $y = \dfrac{x^7}{4} + x^3$

(b) $y = \dfrac{x^7}{4} + x$

(c) $y = \dfrac{x^7}{10} + \dfrac{23}{10}x^{-3}$

(d) $y = \dfrac{x^{10}}{7} + \dfrac{31}{28}$

(e) $y = \dfrac{x^{10}}{7} + x^3$

8. Which of the following is a solution to the initial value problem?

$$\frac{dy}{dt} = \frac{ty^2}{1+t^2}, \qquad y(0) = 1.$$

(a) $\ln(y) = \dfrac{t^3}{3}$

(b) $\dfrac{y^3}{3} = \tan^{-1}(t) + \dfrac{1}{3}$

(c) $y = \dfrac{1}{1 - \frac{1}{2}\ln(1+t^2)}$

(d) $\ln(y) = \tan^{-1}(t)$

(e) $y = \dfrac{1}{1 - \tan^{-1}(t)}$

Part II: Partial credit questions (11 points each). Show your work.

9. (a) Apply the Gram-Schmidt Process to construct an orthogonal basis for the subspace

$$V = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 4 \end{bmatrix} \right\} \text{ of } \mathbb{R}^4.$$

(b) Find an orthonormal basis for V from the orthogonal basis found in part (a).

10. Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

(a) Find the least squares solution to the equation $A\vec{x} = \vec{b}$.

(b) Find the vector in the column space of A which is closest to $\vec{b}$.

11. Consider the differential equation $\left(e^y - \sin(x)\right)dx + \left(xe^y - \dfrac{3}{y}\right)dy = 0.$

(a) Show that the equation is exact.

(b) Find the general implicit solution and express it in the form $f(x, y) = c$.

(c) Find the implicit solution that satisfies the initial condition $y(0) = e$.

12. Willy has a tank containing 10 gallons of milk which initially contains 1 pound of chocolate powder. The well-mixed chocolate milk in the tank is drained at a rate of 3 gallons per hour, and Willy pumps in chocolate milk with a concentration of 1 pound of chocolate powder per gallon at a rate of 3 gallons per hour.

   (a) Set up an initial value problem for the amount $y(t)$ in pounds of chocolate powder in the tank after t hours.

   (b) Solve the initial value problem to find an explicit formula for $y(t)$.