Answer Key 1

**Math 20580** Your Name:____________________

**Final Exam** *May 8, 2007* Instructor's name:____________________

Record your answers to the multiple choice problems by placing an $\times$ through one letter for each problem on this answer sheet. There are 24 multiple choice questions. Each problem counts 6 points and you start with 6 points.
Please sign the honor statement if you agree:

*"I strictly followed the Notre Dame Honor Code during this test."*

Your Signature ____________________

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| 2. | a | b | c | d | • | 14. | • | b | c | d | e |
| 3. | a | b | c | d | • | 15. | a | b | c | • | e |
| 4. | a | b | • | d | e | 16. | a | b | c | • | e |
| 5. | a | b | • | d | e | 17. | a | b | c | d | • |
| 6. | a | b | c | • | e | 18. | a | b | c | • | e |
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| 12. | a | b | • | d | e | 24. | a | • | c | d | e |

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| 1. | a | b | c | d | e | 13. | a | b | c | d | e |
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| 11. | a | b | c | d | e | 23. | a | b | c | d | e |
| 12. | a | b | c | d | e | 24. | a | b | c | d | e |

1. Let $y_1(t)$ and $y_2(t)$ are two fundamental solutions $y'' + y' + \frac{\sin t}{t} y = 0$ with initial conditions $y_1(0) = 1, y_1'(0) = 0$ and $y_2(0) = 0, y_2'(0) = 1$. Then the Wronskian $W(t) = [y_1(t)y_2'(t) - y_1'(t)y_2(t)]$ is equal to

(a) $\frac{\sin t}{t}$. (b) $e^t$. (c) $\frac{1}{t}$. (d) $e^{-t}$. (e) $\sin t$.

2. Let $Y(t) = A_0t^2 + A_1t + A_2$ be a solution to $y'' + 4y = 4t^2$ where $\{A_0, A_1, A_2\}$ are constant numbers. Then $A_2$ is equal to

(a) $-1$ (b) 4 (c) 1 (d) 0 (e) $-\frac{1}{2}$

3. The linear system $\begin{pmatrix} 1 & 5 & -3 \\ 1 & 4 & -1 \\ 2 & 7 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ h \end{pmatrix}$ has a solution if and only if $h =$

(a) 2 (b) 1 (c) 5 (d) 3 (e) $-5$

4. Suppose that $Y(t) = At^s e^{-t} + B$ is a solution to $y'' - 3y' - 4y = -5e^{-t} - 4$, where $\{A, B, s\}$ are constant numbers. Then $A$ is equal to

(a) $-\dfrac{2}{5}$ (b) 4 (c) 1 (d) $-4$ (e) 1

5. Find the adjoint $adj(A)$ of $A = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}$.

(a) $\begin{pmatrix} 7 & 4 \\ 2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -7 & -4 \\ -2 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 7 & -4 \\ -2 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 7 & -2 \\ -4 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} -7 & 4 \\ 2 & -1 \end{pmatrix}$

6. The reduced row echelon form of $\begin{pmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{pmatrix}$ is equal to

(a) $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

7. Find the integrating factor $\mu$ for $dx + (\frac{x}{y} - \sin y + y^2)dy = 0$.

(a) $y$ (b) $\sin y$ (c) 1 (d) $y^2$ (e) $x$

8. Let $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = proj_V \vec{u}$ where $\vec{u} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 7 \end{pmatrix}$ and $V = Span\{\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}\}$. Then $x_1$ is equal to

(a) 2 (b) 4 (c) 1 (d) 0 (e) 3

9. Which of the following sets is an orthonormal basis of $R^2$?

(a) $\{\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \end{pmatrix}\}$

(b) $\{\frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \frac{1}{5}\begin{pmatrix} -4 \\ 3 \end{pmatrix}\}$

(c) $\{\frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \frac{1}{5}\begin{pmatrix} -4 \\ 3 \end{pmatrix}, 0\}$

(d) $\{\frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\}$

(e) $\{\frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \frac{1}{5}\begin{pmatrix} -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$

10. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$. Then $b_{11}$ is equal to

(a) 1 (b) $-2$ (c) 10 (d) $-6$ (e) 5

11. Let $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be a solution to $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 4 & -5 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 9 \end{pmatrix}$. Then $x_1$ is equal to

(a) 3 (b) 16 (c) 8 (d) 9 (e) 29

12. Use the method of reduction of order to find a second solution $y_2 = v(t)y_1(t)$ of the given differential equation $t^2y'' + 2ty' - 2y = 0$ where $y_1(t) = t$. Then $v(t)$ is equal to

(a) $\frac{4}{t}$ (b) $t$ (c) $t^{-3}$ (d) 1 (e) $t^{-2}$

13. Let $r_1$ and $r_2$ be two roots of the characteristic equation for $y'' + 100y = 0$. Then $r_1$ and $r_2$ are

(a) $\pm 10\sqrt{-1}$ (b) $0, 10$ (c) $-100, 0$

(d) $\pm 10$ (e) $-10 \pm 10\sqrt{-1}$

14. If $\det A = 2$ where $A$ is a $4 \times 4$ matrix, then $\det(-2A)$ is

(a) 32 (b) $-4$ (c) $-32$ (d) 16 (e) $-16$

15. The following two solutions form a fundamental set of solutions of linear homogeneous differential equation $2t^2y'' + 3ty' - y = 0$.

(a) $t^{\frac{3}{2}}, t$ (b) $t, t^{-1}$ (c) $t, 1$ (d) $t^{\frac{1}{2}}, t^{-1}$ (e) $t^{\frac{1}{2}}, 0$

16. If $\mathbf{B} = \{(1 \quad 0), (1 \quad 2)\}$ and $\vec{x} = (1 \quad 6)$, then $[\vec{x}]_{\mathbf{B}}$ is equal to

(a) $(1 \quad 6)$ (b) $(1 \quad 0)$ (c) $(3 \quad 2)$ (d) $(-2 \quad 3)$ (e) $(1 \quad 2)$

17. The eigenvalues of $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$ are

(a) $-3, -5, -3$ (b) $1, -5, 0$ (c) $1, 3, 3$ (d) $1, 3, 5$ (e) $1, -2, -2$

18. Let $y(t)$ be the unique solution to the initial value problem $y'' - y = 0$, $y(0) = 2$, $y'(0) = 0$. Then $y(1)$ is equal to

(a) $2e$ (b) $2$ (c) $2e^{-1}$ (d) $e + e^{-1}$ (e) $2e - 2$

19. Let $y(t)$ be the unique solution to $y' + \frac{2}{t}y = 4t$ with initial condition $y(1) = 3$. Then $y(2)$ is equal to

(a) $8$ (b) $\ln 2 + 2$ (c) $4 + \frac{1}{2}$ (d) $e^4 + 2$ (e) $8 + \frac{1}{4}$

20. Let $Y(t) = v_1(t)\cos 3t + v_2(t)\sin 3t$ be a solution to $y'' + 9y = \frac{1}{\sin 3t}$. Then $v_2(t)$ is equal to

(a) $\frac{1}{\sin 3t}$ (b) $\frac{t}{3}$ (c) $\cos 3t$

(d) $\frac{1}{9}\ln|\sin 3t|$ (e) $\frac{1}{3}\ln|\sin 3t|$

21. If $y' = 2y^{100}(3-y)$ and $y(0) = 5$, then find $\lim_{t\to\infty} y(t)$. (Hint: This is an autonomous equation. You can find the answer by studying graphs of the solution).

(a) 2 (b) 3 (c) 1 (d) 0 (e) 5

22. Let $y(t)$ be the unique solution to the initial value problem $y'' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = 0$. Then $y(1)$ is equal to

(a) $e$ (b) $e^{-1}$ (c) $e + e^{-1}$ (d) 1 (e) 0

23. Let $y(t)$ be the unique solution to the equation $y' = y^2$ with $y(0) = -1$. Then $y(1)$ is equal to

(a) 0 (b) $\frac{-1}{2}$ (c) $-4$ (d) $-1$ (e) $-3$

24. The determinant of $\begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & 1 \\ 0 & -2 & 0 \end{pmatrix}$ is equal to

(a) 1 (b) 2 (c) $-2$ (d) 0 (e) 5

**1.** By Abel's Theorem, $W = ce^{-\int p(t)dt} = ce^{-t}$. Evaluating at 0 shows $W = e^{-t}$.

**2.** $2A_0 + 4A_2 + 4A_1 t + 4A_0 t^2 = 4t^2$ so $A_2 = 1$, $A_1 = 0$ and $2A_0 + 4A_2 = 0$ so $A_2 = -1/2$.

**3.** $\begin{bmatrix} 1 & 5 & -3 & | & -4 \\ 1 & 4 & -1 & | & -3 \\ 2 & 7 & 0 & | & h \end{bmatrix}$ $\begin{bmatrix} 1 & 5 & -3 & | & -4 \\ 0 & -1 & 2 & | & 1 \\ 0 & -3 & 6 & | & h+8 \end{bmatrix}$ $\begin{bmatrix} 1 & 5 & -3 & | & -4 \\ 0 & -1 & 2 & | & 1 \\ 0 & 0 & 0 & | & h+5 \end{bmatrix}$

If this system has a solution $h + 5 = 0$.

**4.** $Y'' = As(s-1)t^{s-2}e^{-t} - Ast^{s-1}e^{-t} - Ast^{s-1}e^{-t} + At^s e^{-t}$

$Y' = Ast^{s-1}e^{-t} - At^s e^{-t}$

so $L[Y] = (A+3A-4A)t^s e^{-t} + (-2As-3As+0)t^{s-1}e^{-t} + (s(s-1)A+0+0)t^{s-2}e^{-t} - 4B = -5Ast^{s-1}e^{-t} + s(s-1)At^{s-2}e^{-t} - 4B$. Since $L[Y] = -5e^{-t} - 4$, $s = 1$, $B = 1$ and $A = 1$

**5.** $adj(A) = [a_{ij}]$ where $a_{ij} = (-1)^{i+j}C_{ji}$ and $C_{k\ell}$ is the determinant of the $k-\ell^{\text{th}}$ minor: hence $\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix}$

**6.** $\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix}$ $\begin{bmatrix} 6 & 0 & 12 \\ 3 & -1 & 3 \\ 2 & 1 & 7 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 1 & 7 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

**7.** $\dfrac{\partial M}{\partial y} = 0$, $\dfrac{\partial N}{\partial x} = \dfrac{1}{y}$. Hence $M_y - N_x = -\dfrac{1}{y} = -\dfrac{1}{y}N$. Hence $\mu = e^{\ln y} = y$

**8.** The vectors in $V$ are an orthonormal pair. Hence $proj_V \vec{u} = (\vec{u} \bullet \vec{v}_1)v_1 + (\vec{u} \bullet \vec{v}_2)v_2$.

$\vec{u} \bullet \vec{v}_1 = \dfrac{1}{2}(1+3+1+7) = \dfrac{12}{2} = 6$.

$\vec{u} \bullet \vec{v}_2 = \dfrac{1}{2}(1-3-1+7) = \dfrac{4}{2} = 2$.

Hence $proj_V \vec{u} = \dfrac{1}{2}\begin{pmatrix} 8 \\ 4 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 4 \end{pmatrix}$

**9.** (a) is orthogonal but not unit length; (b) is orthonormal; (c) is not linearly independent; (d) is not a spanning set; (e) is not linearly independent

**10.** $\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 2 & 3 & 2 & | & 0 & 1 & 0 \\ 3 & 8 & 2 & | & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 5 & -1 & | & -3 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & -1 & | & 7 & -5 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -7 & 5 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 & | & 10 & -6 & 1 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -7 & 5 & -1 \end{bmatrix} \text{ and so } b_{11} = 10$$

**11.** $\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 2 & -8 & | & 8 \\ 4 & -5 & -9 & | & 9 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 2 & -8 & | & 8 \\ 0 & 3 & -13 & | & 9 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 1 & -4 & | & 4 \\ 0 & 3 & -13 & | & 9 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 1 & -4 & | & 4 \\ 0 & 0 & -1 & | & -3 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 1 & -4 & | & 4 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 1 & 0 & | & 16 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & 0 & | & -3 \\ 0 & 1 & 0 & | & 16 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & | & 29 \\ 0 & 1 & 0 & | & 16 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$

**12.** $y = tv$, $y' = v + tv'$, $y'' = v' + v' + tv'' = 2v' + tv''$ so $L[tv] = t^3v'' + t^2(2v' + 2v') + t(2v - 2v) = 0$ or $t^3v'' + t^2 4v' = 0$ or $tv'' = -4v'$ or $\dfrac{d}{dt} \ln|v'| = \dfrac{-4}{t}$. Then $\ln|v'| = -4\ln|t| + C$ or $v' = At^{-4}$ and $v = Bt^{-3} + K$. Hence $t^{-3}$ is the answer.

**13.** $r_1$ and $r_2$ are roots of $r^2 + 100r = 0$ so $r = \pm\sqrt{-100} = \pm 10\sqrt{-1}$.

**14.** $\det(cA) = c^n \det A$ if $A$ is $n \times n$ so $\det(-2A) = (-2)^4 \det A = 32$.

**15.** We are looking for solutions of the form $t^s$ so $2s(s-1) + 3s - 1 = 0$ or $2s^2 + s - 1 = 0$. $(2s-1)(s+1) = 0$ so $s = -1$ and $s = 1/2$.

**16.** We are being asked to solve $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. By inspection, second entry is 3 and then first entry is $-2$ so $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Or use Cramer's rule or row reduction.

**17.** To set up and solve the cubic equation is hard and takes a long time. The problem is easy because you have a short list of possible answers. The trace of the matrix is the sum of the eigenvalues with multiplicity. The trace is $1 + (-5) + 1 = -3$ and $1, -2, -2$ is the only one of the answers which sums to $-3$.

**18.** The characteristic equation is $r^2 - 1 = 0$ so $y = ae^t + be^{-t}$ and $y' = ae^t - be^{-t}$. $y(0) = a + b = 2$ and $y'(0) = a - b = 0$. Hence $a = b = 1$ so $y = e^t + e^{-t}$ and $y(1) = e + e^{-1}$.

**19.** $L[t^r] = rt^{r-1} + 2t^{r-1} = (r+2)t^{r-1}$. Hence one particular solution is $t^2$ and a solution to the homogeneous system is $t^{-2}$ so the general solution is $y = t^2 + \dfrac{C}{t^2}$. Hence $y(1) = 1 + C = 3$ so $C = 2$ and $y(2) = 2^2 + \dfrac{2}{2^2} = 4.5$ or $4 + \dfrac{1}{2}$.

**20.** Use Variation of Parameters. The Wronskian is

$$W(t) = \det \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix} = 3$$

One easy way to remember the formulas in the book is the following. A particular solution is given by

$$Y = \det \begin{vmatrix} u_2 & -u_1 \\ y_1 & y_2 \end{vmatrix} = \det \begin{vmatrix} \displaystyle\int \frac{g(t)\, y_1(t)}{W(t)}\, dt & \displaystyle\int \frac{g(t)\, y_2(t)}{W(t)}\, dt \\ y_1 & y_2 \end{vmatrix}$$

In this problem we want

$$\int \frac{g(t)\, y_1(t)}{W(t)}\, dt = \frac{1}{3} \int \frac{\cos(3t)}{\sin(3t)} dt = \frac{1}{3} \cdot \frac{1}{3} \cdot \ln|\sin(3t)|$$

**21.** The solution starts out in the strip above $y = 3$ since $y(0) = 5$ and hence it stays there. In this strip, $y$ is decreasing since $y' < 0$. Hence the limit is 3.

**22.** The characteristic equation is $r^2+2r+1 = 0$ so $r = -1$ is a double root and hence the general solution to the homogeneous equation is $y = ae^{-t} + bte^{-t}$, $y' = -ae^{-t} + be^{-t} - bte^{-t} = (b-a)e^{-t} - bte^{-t}$. Hence $y(0) = a + b = 1$ and $y'(0) = b - a = 0$ so $a = b = \dfrac{1}{2}$. Hence $y = e^{-t}\dfrac{1+t}{2}$ so $y(1) = \dfrac{2}{2e} = \dfrac{1}{e}$.

**23.** The equation is separable so $\displaystyle\int y^{-2} dy = \int dt$ or $-y^{-1} = t + C$ or $y^{-1} = A - t$ or $y = \dfrac{1}{A-t}$. $Y(0) = \dfrac{1}{A} = -1$ so $y = \dfrac{-1}{1+t}$. Hence $y(1) = \dfrac{-1}{2}$.

**24.** Expand along the last column $\det A = 0 - \det \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} + 0 = -(-2) = 2$.