Math 20580 Name:____________________

Final Exam Instructor:____________________

May 8, 2023 Section:____________________

Calculators are NOT allowed. You will be allowed 2 hours to do the test.

There are 20 multiple choice questions worth 7 points each. You will receive 10 points for following the instructions. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. a b c d e
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19. a b c d e
20. a b c d e

1. Consider the bases $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. Find the change of basis matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathcal{P}}$ from $\mathcal{B}$ to $\mathcal{C}$.

(a) $\begin{bmatrix} 2 & 1 \\ 5 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -3 \\ 2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 8 & -1 \\ -5 & 2 \end{bmatrix}$

(e) none of the above

2. Which number is **not** an eigenvalue of the following matrix?

$$\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 4 & 0 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

(a) -3 (b) 4 (c) 1 (d) 2 (e) -5

3. If A is a 4×6 matrix of rank 2, what is the dimension of the orthogonal complement of the row space of A^T?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

4. Consider the differential equation

$$(3x \sin y + 2e^y)dx + (x^2 \cos y + xe^y)dy = 0.$$

Find an integrating factor μ depending only on x that makes the equation exact.

(a) $\mu(x) = e^x$ (b) $\mu(x) = \sin x$ (c) $\mu(x) = x^2$ (d) $\mu(x) = \cos x$ (e) $\mu(x) = x$

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + 2y \\ 3x + 8y \end{bmatrix}.$$

Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that the matrix of T with respect to $\mathcal{B}$ (usually denoted by $[T]_{\mathcal{B}}$ or $\underset{\mathcal{B}\leftarrow\mathcal{B}}{T}$) is a diagonal matrix.

(a) $\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ (c) $\left\{\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}$

(d) $\left\{\begin{bmatrix} 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ (e) $\left\{\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix}\right\}$

6. Which of the following sets of vectors is linearly independent?

(a) $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}\right\}$ (c) $\left\{\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

(d) $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ 2 \\ 0 \end{bmatrix}\right\}$ (e) none of these

7. Let $\mathbb{P}_2$ denote the vector space of polynomials in x of degree at most 2, which has the basis $\mathcal{B} = \{1, x, x^2\}$. Consider the linear transformation

$$T : \mathbb{P}_2 \to \mathbb{P}_2, \qquad T(p(x)) = x^2 p''(x) + xp'(x) + p(x).$$

Compute the matrix of T with respect to $\mathcal{B}$ (denoted $[T]_{\mathcal{B}}$ or $\underset{\mathcal{B}\leftarrow\mathcal{B}}{T}$).

(a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 5 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & -1 \\ 5 & -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 5 \\ 1 & 5 & 0 \end{bmatrix}$

8. Let $\mathcal{B} = \{1 - t, 2 + t^2, t - t^2\}$ be a basis for the vector space $\mathbb{P}_2$ of all polynomials in t of degree at most 2. Find the coordinate vector of $p = 5t - 2t^2$ with respect to $\mathcal{B}$.

(a) $[p]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix}$ (b) $[p]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ (c) $[p]_{\mathcal{B}} = \begin{bmatrix} t - 2 \\ 1 \\ 2 \end{bmatrix}$

(d) $[p]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ (e) $[p]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1/2 \\ 2 \end{bmatrix}$

9. Let

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

If

$$\begin{aligned} r &= \text{dimension of the column space of } A, \\ s &= \text{dimension of the null space of } A, \\ t &= \text{dimension of the row space of } A, \text{ then} \end{aligned}$$

(a) $(r, s, t) = (3, 2, 3)$ (b) $(r, s, t) = (2, 3, 3)$ (c) $(r, s, t) = (2, 3, 2)$
(d) $(r, s, t) = (5, 0, 1)$ (e) $(r, s, t) = (3, 2, 4)$

10. The inverse of the matrix $\begin{bmatrix} -3 & -1 & 2 \\ 4 & 1 & -2 \\ 2 & 0 & -1 \end{bmatrix}$ is

(a) $\begin{bmatrix} -3 & 4 & 2 \\ -1 & 1 & 0 \\ 2 & -2 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 2 \\ -2 & -2 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 2 & 2 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & -2 & -1 \end{bmatrix}$ (e) $\begin{bmatrix} -2 & -1 & 0 \\ 4 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}$

11. Find the solution of the initial value problem

$$\begin{cases} y'' - 5y' - 6y = 0, \\ y(0) = 0, \; y'(0) = 1. \end{cases}$$

(a) $\frac{1}{5}\left(e^{2t} - e^{3t}\right)$ (b) $e^{5t} - e^{6t}$ (c) $\frac{1}{7}\left(e^{6t} - e^{-t}\right)$

(d) $\frac{-1}{5}\left(e^{-2t} - e^{-3t}\right)$ (e) $\frac{-1}{7}\left(e^{t} - e^{-6t}\right)$

12. Compute the Wronskian $W(y_1, y_2)$, where

$$y_1(t) = t\cos(t), \qquad y_2(t) = t\sin(t).$$

(a) 0 (b) t^2 (c) $t\sin(2t)$ (d) $\cos(2t)$ (e) $2t + 1$

13. Consider the differential equation $y'' - 3y' + 2y = e^{2x}$. By the method of undetermined coefficients, a particular solution will have the form

(a) Axe^{2x} (b) $(Ax^2 + B)e^{2x}$ (c) $2e^{2x}$ (d) $Ae^x + Be^{2x}$ (e) Ae^{2x}

14. Find the solution of the initial value problem

$$\begin{cases} y' + 2y = 3e^x, \\ y(1) = 0 \end{cases}$$

(a) $(x-1)e^x$ (b) $e^x - e^{3-2x}$ (c) x^2 (d) e^x (e) $(3e^x - 3)/2$

15. If x is a real number, what are the possible values for the rank of the matrix

$$\begin{bmatrix} 1 & 1 & x \\ 1 & x & 0 \\ x & 0 & 0 \end{bmatrix}?$$

(a) $1, 2$ (b) 0 (c) $0, 1, 2, 3$ (d) $0, 1, 2$ (e) $2, 3$

16. Consider the subspace W of $\mathbb{R}^3$ with basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

Which of the following is the orthogonal projection of $\mathbf{v}$ onto the subspace W?

(a) $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

17. Find a least-squares solution of the equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

(a) $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{-15}{21} \\ \frac{5}{21} \end{bmatrix}$ (c) $\begin{bmatrix} \frac{2}{5} \\ \frac{1}{2} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{21}{25} \\ \frac{-10}{25} \end{bmatrix}$ (e) $\begin{bmatrix} \frac{25}{21} \\ \frac{-10}{21} \end{bmatrix}$

18. The function $y_1(x) = x^2$ is a solution of the differential equation

$$x^2y'' + 2xy' - 6y = 0, \;\; x > 0.$$

Using the method of reduction of order, find a second solution $y_2(x)$ for this equation.

(a) $1/x^3$ (b) e^{3x} (c) $x \ln(x)$ (d) $x^2 + 2x - 6$ (e) x^2e^x

19. Given that $y_1 = t$ and $y_2 = t^{-1}$ form a fundamental set of solutions for the differential equation $y'' + t^{-1}y' - t^{-2}y = 0$, the variation of parameters method gives a particular solution Y to the corresponding nonhomogeneous equation

$$y'' + t^{-1}y' - t^{-2}y = t^{-1}, \quad t > 0,$$

of the form

$$Y = tu_1(t) + t^{-1}u_2(t).$$

Which of the following functions can be $u_1(t)$?

(a) $u_1(t) = \frac{1}{2}\ln t$ (b) $u_1(t) = t\ln t$ (c) $u_1(t) = -\frac{1}{2}t^2$
(d) $u_1(t) = t^2\ln t$ (e) $u_1(t) = 2t + 5t^{-1}$

20. Which formula describes implicitly the solution of the initial value problem

$$\frac{dy}{dx} = \frac{x+1}{x \cdot (y^2+1)}, \quad y(1) = 0, \qquad x > 0.$$

(a) $3x + \ln x - 3y = 3$ (b) $y^3 + 3y = 3x + 3\ln x - 3$ (c) $x - \ln x - y^3 = 1$
(d) $y^3 + y = 3x - 3$ (e) $-y^3 + 3y = 3x + 3\ln x$