Name: Soln. Date: 04/11/2024

Math 20580 Tutorial
Worksheet 10

1. For each of the following, solve the differential equation.
   (a) $\frac{dx}{dt} = 4(x^2+1)$,
   (b) $x^2\frac{dy}{dx} = y - yx$.

(a) $\frac{1}{x^2+1}\,dx = 4\,dt$

$$\int \frac{1}{x^2+1}\,dx = \int 4\,dt$$

$$\arctan(x) = 4t + C$$

$$x = \tan(4t+c)$$

(b) $x^2\,dy = y(1-x)\,dx$

$$\frac{1}{y}\,dy = \frac{1-x}{x^2}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x^2}\,dx - \int \frac{1}{x}\,dx$$

$$\ln y = -\frac{1}{x} - \ln x + C$$

$$y = e^{-\frac{1}{x} - \ln x + c}$$

or $y = \dfrac{ce^{-\frac{1}{x}}}{x}$

2. Solve the differential equation $y' + 3\sqrt{t}y = \sqrt{t}$.

Integrating Factor: $\int 3\sqrt{t}\,dt = 3 \cdot \frac{2}{3} t^{3/2}$

$$\Rightarrow \mu = e^{2t^{3/2}}$$

$$y'\mu + 3\sqrt{t}y\mu = \sqrt{t}\mu$$

$$(y\mu)' = \sqrt{t}\mu$$

$$y\mu = \int \sqrt{t} e^{2t^{3/2}} dt$$

Let $u = 2t^{3/2}$, then $du = 3\sqrt{t}\,dt$

$$RHS = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + c = \frac{e^{2t^{3/2}}}{3} + C$$

$$\Rightarrow y = \frac{\frac{e^{2t^{3/2}}}{3} + C}{e^{2t^{3/2}}} = \frac{1}{3} + Ce^{-2t^{3/2}}$$

3. Find the solution to the initial value problem

$$t\frac{dy}{dt} + y = t\sin t, \quad y(\pi) = 1.$$

Then, find the maximal interval of existence of the solution.

$$\frac{dy}{dt} + \frac{1}{t}y = \sin t$$

Integrating Factor: $\mu = e^{\int \frac{1}{t}dt} = e^{\ln t} = t.$

$$yt = \int \sin t \cdot t\, dt = -t\cos t + \sin t + C$$

$$\Rightarrow y = -\cos t + \frac{\sin t}{t} + \frac{C}{t}$$

$$1 = -\cos(\pi) + \frac{\sin(\pi)}{\pi} + \frac{C}{\pi}$$

$$\Rightarrow 1 = 1 + 0 + \frac{C}{\pi}$$

$$\Rightarrow C = 0$$

Thus, $y = -\cos t + \frac{\sin t}{t}$

Interval: $(0, +\infty)$

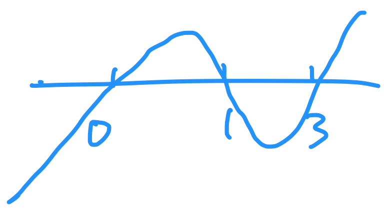
Name: Date: 04/11/2024

4. Find all the *stable* equilibrium solutions of the autonomous system

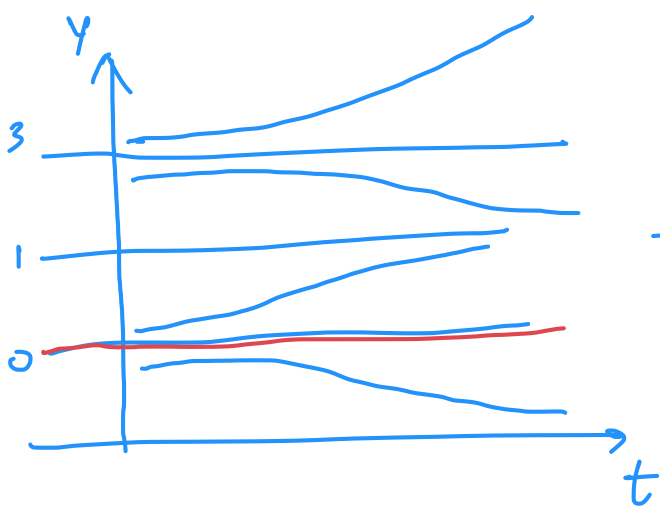
$$\frac{dy}{dt} = 3y - 4y^2 + y^3$$

Compute $\lim_{t->-\infty} y(t)$ where $y(t)$ is the solution of the ODE satisfying the initial condition y(0)=0.

$$3y - 4y^2 + y^3 = y(3 - 4y + y^2) = y(y-1)(y-3)$$



For a stable equilibrium at y_0, $\frac{dy}{dt}$ changes sign from + to − . Thus, $y=1$ is a stable equilibrium soln.



$y=0$ satisfies $y(0)=0$ and $y=0$ is an equilibrium soln.

Thus, $\lim_{t\to-\infty} y(t) = 0$.