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# Math 20580 Tutorial
## Worksheet 11

1.

We will analyze the simple (or undamped) harmonic oscillator. By following these steps, we will verify that

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

is the general solution to the homogeneous differential equation

$$\frac{d^2}{dt^2}x + \omega^2 x = 0$$

on the interval $(-\infty, \infty)$.

i. First use calculus to verify that $\cos \omega t$ and $\sin \omega t$ both satisfy the 2nd-order differential equation.

ii. Next use the Wronskian, $\det \begin{pmatrix} \cos \omega t & \sin \omega t \\ \frac{d}{dt}\cos \omega t & \frac{d}{dt}\sin \omega t \end{pmatrix}$, to verify that these two solutions are linearly independent on the whole interval.

iii. Lastly, use the superposition principle to write the general solution to the differential equation as a linear combination of the solutions.

i) $\frac{d}{dt}\cos \omega t = -\omega \sin \omega t$

$\frac{d^2}{dt^2}\cos \omega t = -\omega^2 \cos \omega t$

$\frac{d^2}{dt^2}\cos \omega t + \omega^2 \cos \omega t = -\omega^2 \cos \omega t + \omega^2 \cos \omega t = 0$

$\frac{d}{dt}\sin \omega t = \omega \cos \omega t$

$\frac{d^2}{dt^2}\sin \omega t = -\omega^2 \sin \omega t$

$\frac{d^2}{dt^2}\sin \omega t + \omega^2 \sin \omega t = 0$

ii) $\det \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\omega \sin \omega t & \omega \cos \omega t \end{pmatrix} = \omega \cos^2 \omega t + \omega \sin^2 \omega t$

$= \omega(\cos^2 \omega t + \sin^2 \omega t)$

$= \omega \cdot 1$

$\neq 0$, for all $t \in \mathbb{R}$

So $\cos \omega t$ and $\sin \omega t$ are linearly independent

iii) Superposition principle: General solution is

$X = C_1 \cos \omega t + C_2 \sin \omega t$

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2.

Find particular solutions to the inhomogeneous differential equations by inspection.

1. $y'' + 2y = 10$
2. $y'' + 2y = -4x$
3. $y'' + 2y = 10 - 4x$

1) Guess $y = c$

Check
$y' = 0$
$y'' = 0$

so $y'' + 2y = 0 + 2c = 10$
$\Rightarrow c = 5$
$\Rightarrow y = 5$ is particular solution

2) Guess $y = cx$

Check
$y' = c$
$y'' = 0$

so $y'' + 2y = 0 + 2cx = -4x$
so $c = -2$
$\Rightarrow$ $y = -2x$ is particular solution

3) Superposition principle:
$y = 5 - 2x$ is particular solution

3.

We can use *reduction of order* to find general solutions for inhomogeneous or homogeneous 2nd-order linear differential equations, provided one solution to the associated homogeneous equation is already known. We will use this to analyze a critically damped harmonic oscillator. That is, we consider the differential equation,

$$x'' + 2x' + x = 0,$$

where one solution $x_1$ is already known, and we follow the steps to find the general solution $c_1x_1 + c_2x_2$, by first assuming the solution $x_2$ is of the form $x_2 = ux_1$ for an unknown function $u$.

i Verify that $x_1 = e^{-t}$ solves the homogeneous equation $x'' + 2x' + x = 0$.

ii Let $x_2 = ux_1$ for an unknown function $u(t)$, differentiate $x_2$ (twice) and substitute $x_2, x_2'$, and $x_2''$ into the original differential equation to obtain a new differential equation for $u$.

iii Let $w = \frac{du}{dt}$ and apply separation of variables to find $w$.

iv Since $u' = w$, we can apply separation of variables one more time to find $u$.

v Remember that $x_2 = ux_1$ to find $x_2$. Remember that the general solution is given by linear combinations of $x_1$ and $x_2$.

vi What is $\lim_{t\to\infty} x(t)$?

i) $\frac{d^2}{dt^2}e^{-t} + 2\frac{d}{dt}e^{-t} + e^{-t} = e^{-t} + 2(-e^{-t}) + e^{-t} = 0$

ii) $X_2 = ue^{-t}$, $X_2' = u'e^{-t} - ue^{-t} = (u'-u)e^{-t}$

$X_2'' = (u''-u')e^{-t} - (u'-u)e^{-t}$

$= (u''-2u'+u)e^{-t}$

$0 = X_2'' + 2X_2' + X_2 = (u''-2u'+u)e^{-t} + 2(u'-u)e^{-t} + ue^{-t}$

$= (u''-2u'+u+2u'-2u+u)e^{-t}$

$= u''e^{-t}$

$\Rightarrow u'' = 0$, since $e^{-t} \neq 0$

iii) let $\omega = u'$, then $\omega' = 0$, so $\omega = c_1$, $c_1 \in \mathbb{R}$

iv) $u' = c_1$, so $u = c_1t + c_2$, $c_1, c_2 \in \mathbb{R}$

v) $X_2 = uX_1 = (c_1t + c_2)e^{-t} = c_1te^{-t} + c_2e^{-t}$

$X = c_1te^{-t} + c_2e^{-t}$ is general solution, since $X_1 = e^{-t}$ already appears in $X_2$

vi) $\lim_{t\to\infty}(c_1te^{-t} + c_2e^{-t}) = 0$

4.

Use the auxiliary equation to solve the homogeneous constant-coefficient linear ODE with boundary values:

$$y'' - 10y' + 25y = 0$$
$$y(0) = 0$$
$$y(1) = 0.$$

Recall that we find the auxiliary equation by replacing the functions $y^{(n)}$ with the variables $s^n$, and solving the resulting polynomial equation for $s$, assuming that our general solution will look like $y = e^{sx}$. Note that some case-by-case analysis is usually required to obtain the general solution from the list of roots.

auxiliary equation : $s^2 - 10s + 25 = 0$

$s^2 - 10s + 25 = (s-5)^2 = 0$

So we have real <u>repeated</u> roots $s=5,\ s=5$

So one solution is $y_1 = e^{5x}$ and we need <u>reduction of order</u> (or memorizing patterns and lucky guesses) to find the other.

Let $y_2 = u y_1 = u e^{5x}$, then $y_2' = u'e^{5x} + 5ue^{5x} = (u'+5u)e^{5x}$

$y_2'' = (u''+5u')e^{5x} + 5(u'+5u)e^{5x}$

$= (u'' + 10u' + 25u)e^{5x}$

Make

$0 = y_2'' - 10y_2' + 25y_2$

$= (u'' + 10u' + 25u)e^{5x} - 10(u'+5u)e^{5x} + 25ue^{5x}$

$= (u'' + 10u' + 25u - 10u' - 50u + 25u)e^{5x}$

$= (u'')e^{5x}$. Since $e^{5x} \neq 0$, we have $u'' = 0$

Now by integrating twice, $u = c_1 t + c_2$

So $y_2 = c_1 t e^{5t} + c_2 e^{5t}$ and general solution is just $y = c_1 t e^{5t} + c_2 e^{5t}$

$0 = y(0) = 0 + c_2 \Rightarrow c_2 = 0$

$0 = y(1) = c_1 e^{5t} \Rightarrow c_1 = 0$

So only solution to BVP is $y = 0$.