Math 20580 Tutorial Week 01 Name ______________

Section ______________

Tutorial Worksheet

1. Find all solutions to the linear system by following the given steps

$$\begin{cases} x + 2y + 3z = 2 \\ 2x + 3y + z = 4 \\ y + z = 8 \end{cases}.$$

Augmented Matrix

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 1 & 4 \\ 0 & 1 & 1 & 8 \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -5 & 0 \\ 0 & 1 & 1 & 8 \end{bmatrix}.$$

$$\xrightarrow{R_3 \mapsto R_2 + R_3} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -5 & 0 \\ 0 & 0 & -4 & 8 \end{bmatrix} \xrightarrow[R_3 \mapsto -\frac{1}{4}R_3]{R_2 \mapsto -R_2} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

(echelon form)

$$\xrightarrow{R_1 \mapsto R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow[R_2 \mapsto R_2 - 5R_3]{R_1 \mapsto R_1 + 7R_3} \begin{bmatrix} 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

(reduced echelon form).

$$\Rightarrow \begin{cases} x = -12 \\ y = 10 \\ z = -2 \end{cases}$$

2. As a generalization to the first question, can you find the solution to the system

$$\begin{cases} x + 2y + 3z = a \\ 2x + 3y + z = b \\ y + z = c \end{cases}$$

for arbitrary real numbers a, b, and c?

$$\begin{bmatrix} 1 & 2 & 3 & a \\ 2 & 3 & 1 & b \\ 0 & 1 & 1 & c \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & a \\ 0 & -1 & -5 & b-2a \\ 0 & 1 & 1 & c \end{bmatrix}$$

$$\xrightarrow{R_3 \mapsto R_2 + R_3} \begin{bmatrix} 1 & 2 & 3 & a \\ 0 & -1 & -5 & b-2a \\ 0 & 0 & -4 & c+b-2a \end{bmatrix} \xrightarrow[R_3 \mapsto -\frac{1}{4}R_3]{R_2 \mapsto -R_2} \begin{bmatrix} 1 & 2 & 3 & a \\ 0 & 1 & 5 & 2a-b \\ 0 & 0 & 1 & -\frac{1}{4}(c+b-2a) \end{bmatrix}$$

$$\xrightarrow{R_1 \mapsto R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -7 & -3a+2b \\ 0 & 1 & 5 & 2a-b \\ 0 & 0 & 1 & -\frac{1}{4}(c+b-2a) \end{bmatrix} \xrightarrow[R_2 \mapsto R_2 - 5R_3]{R_1 \mapsto R_1 + 7R_3} \begin{bmatrix} 1 & 0 & 0 & \frac{a}{2} + \frac{b}{4} - \frac{7}{4}c \\ 0 & 1 & 0 & -\frac{a}{2} + \frac{b}{4} + \frac{5}{4}c \\ 0 & 0 & 1 & \frac{a}{2} - \frac{b}{4} - \frac{c}{4} \end{bmatrix}$$

$$\Rightarrow \begin{cases} x = \frac{a}{2} + \frac{b}{4} - \frac{7}{4}c \\ y = -\frac{a}{2} + \frac{b}{4} + \frac{5}{4}c \\ z = \frac{a}{2} - \frac{b}{4} - \frac{c}{4} \end{cases}$$

3. Which matrices below are in echelon form? Which are in reduced echelon form?

$$A = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \cancel{D = \begin{bmatrix} 1 & 2 & - \\ 0 & 1 & 0 \end{bmatrix}}$$

$$D = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Echelon form : all

Reduced Echelon form A. C

4. Which columns of the matrix below are pivot and which are free?

$$\begin{bmatrix} 1 & 2 & -1 & -3 & 2 \\ 2 & 5 & -1 & -6 & 6 \\ 3 & 7 & -2 & -8 & 8 \end{bmatrix}$$

Remark: A *free column* is a column which corresponds to a free variable.

$$\begin{bmatrix} 1 & 2 & -1 & -3 & 2 \\ 2 & 5 & -1 & -6 & 6 \\ 3 & 7 & -2 & -8 & 8 \end{bmatrix} \xrightarrow[R_3 \mapsto R_3 - 3R_1]{R_2 \mapsto R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & -3 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \mapsto R_3 - R_2} \begin{array}{c} \begin{array}{ccccc} \phantom{\boxed{1}} & & \text{free} & & \end{array} \\ \begin{bmatrix} \boxed{1} & 2 & -1 & -3 & 2 \\ 0 & \boxed{1} & 1 & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{bmatrix} \end{array} \quad \text{(echelon form).}$$

pivot.

5. Find all solutions to the linear system

$$\begin{cases} x - 3y + z - w = 2 \\ 2x - 6y + 3z - w = 3 \\ 3x - 9y + 5z - w = 4 \end{cases}.$$

$$\begin{bmatrix} 1 & -3 & 1 & -1 & 2 \\ 2 & -6 & 3 & -1 & 3 \\ 3 & -9 & 5 & -1 & 4 \end{bmatrix} \xrightarrow[R_3 \mapsto R_3 - 3R_1]{R_2 \mapsto R_2 - 2R_1} \begin{bmatrix} 1 & -3 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 2 & -2 \end{bmatrix}.$$

free

$$\xrightarrow[R_3 \mapsto R_3 - 2R_2]{R_1 \mapsto R_1 - R_2} \begin{bmatrix} \boxed{1} & -3 & 0 & -2 & 3 \\ 0 & 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

pivot

$$\Rightarrow \begin{cases} x - 3y - 2w = 3 \Rightarrow x = 3 + 3y + 2w. \\ \cancel{y \text{ is free}} \\ z + w = -1 \Rightarrow z = -1 - w \\ y \text{ is free} \\ w \text{ is free} \end{cases}$$

$$\Rightarrow \begin{cases} x = 3 + 3y + 2w \\ y \text{ is free} \\ z = -1 - w \\ w \text{ is free} \end{cases}$$