

**M20580 L.A. and D.E. Tutorial
Worksheet 2**

1. Use either Gaussian or Gauss-Jordan elimination algorithm to solve the following linear systems. What is the rank of each?

$$\text{a) } \begin{cases} x_1 + 2x_2 - 3x_3 = 9 \\ 2x_1 - x_2 + x_3 = 0 \\ 4x_1 - x_2 + x_3 = 4 \end{cases} \qquad \text{b) } \begin{cases} x_1 - 3x_2 - 2x_3 = 0 \\ -x_1 + 2x_2 + x_3 = 0 \\ 2x_1 + 4x_2 + 6x_3 = 0 \end{cases}$$

Solution:

- a) Let's row reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{array} \right] \xrightarrow{R_2-2R_1, R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 0 & -9 & 13 & -32 \end{array} \right] \xrightarrow{-1/5 R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & 1 & -7/5 & 18/5 \\ 0 & -9 & 13 & -32 \end{array} \right] \xrightarrow{R_3+9R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & 1 & -7/5 & 18/5 \\ 0 & 0 & 2/5 & 2/5 \end{array} \right] \rightarrow \begin{cases} x_1 + 2x_2 - 3x_3 = 9 \\ x_2 - 7/5x_3 = 18/5 \\ 2/5x_3 = 2/5 \end{cases}$$

We see from the REF that the rank is three. Now use back substitution to solve the system:

$$x_3 = 1, \rightarrow x_2 = \frac{18+7}{5} = 5, \rightarrow x_1 = 9 - 10 + 3 = 2.$$

Thus the solution of the first system is $\mathbf{x} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$.

- b) As in the previous bullet, row reduce the matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 4 & 6 & 0 \end{array} \right] \xrightarrow{R_2+R_1, R_3-2R_1} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 10 & 10 & 0 \end{array} \right] \xrightarrow{R_3+10R_2} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \begin{cases} x_1 - 3x_2 - 2x_3 = 0 \\ -x_2 - x_3 = 0 \end{cases}$$

Now we see from the REF that the rank is two. Let $t := x_3$ be a parameter. Using back substitution, we find that $x_2 = -t$, $x_1 = -3t + 2t = -t$. Thus the

solution is $\mathbf{x} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

2. Given the augmented matrices of some linear systems, determine how many solutions they have, if any; also, if the corresponding linear system is consistent, determine its rank and the number of free variables.

$$\text{a) } \left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & -1 \\ 0 & 1 & -2 & 2 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -2 & 2 \end{array} \right]$$

$$\text{b) } \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{c) } \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & -2 & 2 & 3 & 0 \\ 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 & -1 \end{array} \right]$$

Solution:

- a) The matrix is not yet in a REF, but after $R_4 - 2R_2$ it is in a REF:

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & -1 \\ 0 & 1 & -2 & 2 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -2 & 2 \end{array} \right] \xrightarrow{R_4 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & -1 \\ 0 & 1 & -2 & 2 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

We see that the system is consistent. The rank of the system is 3 and the number of variables is 4, hence one of the variables must be a free variable. This implies the system has infinitely many solutions.

- b) As in the previous case, we need to row reduce the matrix first (you can also spot an issue in this example without any computation):

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

The third row represents the equation $0 = -3$, which is a contradiction. Therefore, the system is inconsistent.

- c) We row reduce the matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & -2 & 2 & 3 & 0 \\ 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 & -1 \end{array} \right] \xrightarrow{R_4 + R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & -2 & 2 & 3 & 0 \\ 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right]$$

The system is consistent. The rank is 4 and the number of variables is 4, hence there are no free variables, and thus the solution is unique.

3. Determine if a given vector \mathbf{v} is a linear combination of vectors \mathbf{u}_i . If yes, write \mathbf{v} as a linear combination of those vectors.

$$\text{a) } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \quad \text{b) } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Hint: a vector \mathbf{v} is a linear combination of \mathbf{u}_1 and \mathbf{u}_2 if you can find scalars α_1 and α_2 such that $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2$. The question about existence of a linear combination can be translated into the question about consistency of a certain linear system. The columns of the matrix of the system are \mathbf{u}_1 and \mathbf{u}_2 , and it's augmented by \mathbf{v} . If one solves the system, one obtains the scalars α_1 and α_2 .

Solution:

- a) We row reduce the augmented matrix of the system that answers the question:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{array} \right] \xrightarrow{R_2-R_1, R_3-R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{array} \right].$$

We see that the last row corresponds to the equation $0 = 2$, hence the system is not consistent. In terms of vectors, this means that \mathbf{v} is not a linear combination of \mathbf{u}_1 and \mathbf{u}_2 .

- b) This is almost the same problem except we have one extra vector at our disposal. Row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{array} \right] \xrightarrow{R_2-R_1, R_3-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 2 \end{array} \right]$$

The system is consistent, hence \mathbf{v} is a linear combination of \mathbf{u}_1 , \mathbf{u}_2 and \mathbf{u}_3 . To find the actual coefficients, we proceed with back substitution:

$$\alpha_3 = 1, \rightarrow \alpha_2 = 1 + \alpha_3 = 2, \rightarrow \alpha_1 = 1 - \alpha_3 = 0.$$

Thus $\mathbf{v} = 2\mathbf{u}_2 + \mathbf{u}_3$.

4. Given the augmented matrix A of some linear system below, describe how many solutions the system has depending on the values of h and k :

$$A = \left[\begin{array}{ccc|c} 1 & -1 & 0 & h \\ 2 & -1 & -2 & k \\ 0 & 1 & 1 & 1 \end{array} \right]$$

Hint: first, row reduce the matrix to find its REF and determine its rank.

Solution: Row reduce the matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & h \\ 2 & -1 & -2 & k \\ 0 & 1 & 1 & 1 \end{array} \right] \xrightarrow{R_2-2R_1, R_3-R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & h \\ 0 & 1 & -2 & k-2h \\ 0 & 0 & 3 & 1-k+2h \end{array} \right]$$

We see that the rank of the system is three and the number of variables is three. When these two numbers coincide, the system is consistent for any possible augmentation (right-hand side of the system), and the solution is unique.