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M20580 L.A. and D.E. Tutorial
Worksheet 3

1. For the following transformations, use the definition of a linear transformation to either prove it is a linear transformation or give a counterexample to show it is not.

$$\text{(a) } T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x^2 \end{bmatrix} \qquad \text{(b) } T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x+2y \\ 3x-4y \end{bmatrix} \qquad \text{(c) } T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y-1 \end{bmatrix}$$

(a) Not linear. Let $c \in \mathbb{R}$

$$\Rightarrow T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} cy \\ c^2x^2 \end{bmatrix} \neq c\begin{bmatrix} y \\ x^2 \end{bmatrix} = cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

(b) Linear. Let $c \in \mathbb{R}$

$$\Rightarrow \begin{cases} T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -cy \\ cx+2cy \\ 3cx-4cy \end{bmatrix} = c\begin{bmatrix} -y \\ x+2y \\ 3x+4y \end{bmatrix} = cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = \begin{bmatrix} -y_1-y_2 \\ (x_1+x_2)+2(y_1+y_2) \\ 3(x_1+x_2)-4(y_1+y_2) \end{bmatrix} = \begin{bmatrix} -y_1 \\ x_1+2y_1 \\ 3x_1-4y_1 \end{bmatrix} + \begin{bmatrix} -y_2 \\ x_2+2y_2 \\ 3x_2-4y_2 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) \end{cases}$$

(c) Not linear. Let $c \in \mathbb{R}$

$$\Rightarrow T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} cx+1 \\ cy-1 \end{bmatrix} \neq \begin{bmatrix} cx+c \\ cy-c \end{bmatrix} = c\begin{bmatrix} x+1 \\ y-1 \end{bmatrix} = cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

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2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformations defined by:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ -x \end{bmatrix} \qquad \text{and} \qquad S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 3y \\ 2x + y \\ x - y \end{bmatrix}$$

Find the standard matrix for $S \circ T$, i.e. find the matrix A such that $(S \circ T)(\mathbf{u}) = A\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^2$. Determine whether it is possible to find such a matrix for $T \circ S$ and either find the matrix or explain why it does not exist.

Let M_T and M_S be the standard matrices for S and T

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow M_T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow M_S = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow A = M_S M_T = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

A matrix for $T \circ S$ does not exist because S returns a vector in $\mathbb{R}^3$ and the input to T is a vector in $\mathbb{R}^2$ so $T \circ S$ is not a valid transformation.

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3. Determine if the following sets of vectors are linearly independent or not. Also, find the dimension and give a basis of the subspace they span.

$$\text{(a)} \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix} \qquad \text{(b)} \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}3\\2\\-1\end{bmatrix} \qquad \text{(c)} \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}8\\16\end{bmatrix}$$

(a)
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \Rightarrow$$ inconsistent so linearly independent

Since they are linearly independent, the dimension of the subspace is 3 and the 3 vectors form a basis of it.

(b) $3\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}3\\2\\-1\end{bmatrix} \Rightarrow$ linearly dependent

Since the first two vectors are clearly independent, the dimension of the subspace is 2 and $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ forms a basis of it

(c) $-8\begin{bmatrix}1\\0\end{bmatrix} + 16\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}8\\16\end{bmatrix} \Rightarrow$ linearly dependent

Since the first two vectors are clearly independent, the dimension of the subspace is 2 and

$\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ forms a basis of it

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4. Consider the conditions below and the subset of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ which they describe. In each case, either prove the subset of vectors forms a subspace of $\mathbb{R}^2$ or give a counterexample to show it does not.

(a) $x \geq 0, y \geq 0$ (b) $x = 0$ (c) $xy \geq 0$

(a) Not a subspace. Let $x>0, y>0$

$\Rightarrow -1 \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$ where $-x<0, -y<0$

$\Rightarrow$ not closed under scalar multiplication

(b) It is a subspace.

i) $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x=0 \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ belongs to subset

ii) $\begin{pmatrix} 0 \\ y_1 \end{pmatrix} + \begin{pmatrix} 0 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ y_1+y_2 \end{pmatrix} \Rightarrow x=0 \Rightarrow$ closed under addition

iii) $c\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ cy \end{pmatrix} \Rightarrow x=0 \Rightarrow$ closed under multiplication

(c) Not a subspace. Consider $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ both belong to the subset but their sum does not

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -2x + z \\ 5x \\ x - 3z \end{bmatrix}$$

(a) Find the standard matrix of T.

(b) Find a basis for the column space.

(c) Find a basis for the null space.

(a) Let A be the standard matrix. Then

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} -2 & 0 & 1 \\ 5 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$$

(b) Since the middle column is the zero vector and the other columns are not multiples of each other, we have

$$\left\{ \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \right\}$$ forms a basis for the column space of A

(c) $\left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 5 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 \end{array}\right]$

$\xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - 5R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right]$

$\xrightarrow{R_2 \to \frac{1}{15} R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right]$

$\xrightarrow[R_3 \to R_3 + 4R_2]{R_1 \to R_1 + 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Therefore there are infinite solutions of the form

$c\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ c \in \mathbb{R} \Rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ forms a basis of the null space