Name: Solution Date: 02/07/2024

Math 20580 Tutorial
Worksheet 3

1. (a) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}$. Compute $A^T A + 6B^{-1}$.

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} + 6 \cdot \frac{1}{2} \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 14 \\ 14 & 21 \end{bmatrix} + \begin{bmatrix} 6 & 12 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 26 \\ 14 & 24 \end{bmatrix}$$

(b) If the matrices A, B are such that

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad AB = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix}$$

then what is the matrix B?

$$B = A^{-1} \cdot AB$$

$$= \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

2. Use Guass-Jordan method to find the inverse of the given matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 & 1 & 3 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -1 & -4 \\ 0 & 1 & 0 & -3 & 1 & 3 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -1 & -4 \\ 0 & 1 & 0 & -3 & 1 & 3 \\ 0 & 0 & 1 & 3 & -1 & -2 \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} 4 & -1 & -4 \\ -3 & 1 & 3 \\ 3 & -1 & -2 \end{bmatrix}$

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3. Let $\mathbf{b}_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$; and $\mathbf{c}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ be two bases for $\mathbb{R}^2$. Find $P_{\mathcal{C} \leftarrow \mathcal{B}}$.

Method 1.

$$\begin{array}{cc} C & B \\ \left[\begin{array}{cc|cc} 3 & 1 & 5 & 2 \\ 2 & 1 & 4 & 3 \end{array}\right] \end{array} \xrightarrow{R_1 = -R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 2 & 1 & 4 & 3 \end{array}\right]$$

$$\xrightarrow{R_2 = -2R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 5 \end{array}\right]$$

Thus, $P_{C \leftarrow B} = \begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix}$

Method 2.

$$P_{C \leftarrow B} = P_{C \leftarrow E} P_{E \leftarrow B}$$

$P_{E \leftarrow B} = \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}$. $P_{E \leftarrow C} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

$$P_{C \leftarrow E} = (P_{E \leftarrow C})^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Then, $P_{C \leftarrow B} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix}$

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4.

(a) Suppose that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ are two bases for a vector space V. Also suppose that the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is given as

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}.$$

For $\mathbf{v} = 2\mathbf{b}_1 + \mathbf{b}_2$, what is $[\mathbf{v}]_C$, the $\mathcal{C}$-coordinates for $\mathbf{v}$?

$$[v]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$[v]_C = P_{C\leftarrow B} \cdot [v]_B$$

$$= \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

(b) Find the standard coordinates for $\mathcal{C}$ if $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}$.

i.e. Find $P_{E\leftarrow C}$

$$P_{E\leftarrow C} = P_{E\leftarrow B} \cdot P_{B\leftarrow C}$$

$$= \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ -9 & 7 \end{bmatrix}$$

Thus, $c_1 = \begin{bmatrix} -1 \\ -9 \end{bmatrix}$

$c_2 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

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5. Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\}$ for $\mathbb{R}^3$.

   (a) If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, find $\mathbf{x}$ (its coordinate representation in the standard basis).

   (b) What is $P_{\mathcal{B} \leftarrow \mathcal{E}}$ where $\mathcal{E}$ is the standard basis?

(a)

$$[x]_E = P_{E \leftarrow B} [x]_B$$

$$= \begin{bmatrix} 1 & 3 & 1 \\ 1 & 4 & 3 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 10 \\ -1 \end{bmatrix}$$

(b) $P_{B \leftarrow E} = (P_{E \leftarrow B})^{-1}$

Use Gauss-Jordan elimination, we can figure out

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 4 & 3 \\ 1 & 3 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 9 & -3 & -5 \\ -3 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$