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Math 20580 Tutorial
Worksheet 5

1. Show that the following sets (with operations) are subspaces of the specified vector spaces
   - $\mathcal{P}_2 = \{p(x) = a + bx + cx^2 \mid a, b, c \in \mathbb{R}\} \subset \text{Fun}(\mathbb{R})$, the set of polynomials with degree less than or equal to 2, and the usual addition and scalar multiplication, as a subset of the vector space of real-valued functions.
   - $\{A \mid A^T = -A\} \subset \text{Mat}_n$, the set of anti-symmetric $n \times n$ matrices with the usual addition and scalar multiplication, as a subset of the vector space of $n \times n$ real matrices.

$(a_1 + b_1 x + c_1 x^2) + (a_2 + b_2 x + c_2 x^2)$

$= (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2$ ✓

$k(a + bx + cx^2) = (ka) + (kb)x + (kc)x^2$ ✓

Let $A, B \in Mat_n$, s.t. $A^T = -A$, $B^T = -B$.

$(A+B)^T = A^T + B^T$, $-^T$ is linear

$= -A - B$, assumption

$= -(A+B)$, distribution ✓

$(kA)^T = kA^T$, $-^T$ is linear

$= -kA$ ✓

2. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation $T(p(x)) = x\frac{d}{dx}(p(x))$.
   Describe $\ker T$, $\operatorname{range} T$.
   (Hint: consider $T(1), T(x), T(x^2)$ individually, and use linearity.)

$$T(1) = x\frac{d}{dx}(1) = 0$$

$$T(x) = x\frac{d}{dx}(x) = x \cdot 1 = x$$

$$T(x^2) = x\frac{d}{dx}(x^2) = x \cdot (2x) = 2x^2$$

so, $T(a+bx+cx^2) = T(a) + bT(x) + cT(x^2)$
$= bx + 2cx^2$

$\ker T = \{p(x) = a+bx+cx^2 \mid T(p(x)) = 0\}$
$= \{p(x) = a \mid a \in \mathbb{R}\}$

$\operatorname{range} T = \{p(x) = a+bx+cx^2 \mid$ there is some $q(x) = d+ex+fx^2$ and $T(q(x)) = p(x)\}$
$= \{p(x) = bx + 2cx^2 \mid b, c \in \mathbb{R}\}$
$= \{p(x) = bx + cx^2 \mid b, c \in \mathbb{R}\}$

Note $\dim(\ker T) + \dim(\operatorname{range} T) = 1 + 2 = 3$
$= \dim(\mathcal{P}_2)$

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3. Let $T : \mathrm{Mat}_2 \to \mathbb{R}$ be the linear transformation defined in standard coordinates by $T(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a+d$.

   Describe $\ker T$, $\operatorname{range} T$.

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1, \quad T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 0,$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = 0, \quad T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$$

$$T\left(a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = a+d$$

$$\ker T = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a+d = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}$$

$$\operatorname{range} T = \left\{ k \in \mathbb{R} \,\middle|\, \begin{array}{l} \text{there is some } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \text{w/ } T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = k \end{array} \right\}$$

$$= \left\{ k \in \mathbb{R} \,\middle|\, \begin{array}{l} \text{there's a } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \text{s.t. } a+d = k \end{array} \right\}$$

$$= \mathbb{R}, \text{ since } T\left(\begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}\right) = k \text{ for any } k$$

$$\left(\begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} \text{ chosen arbitrarily}\right)$$

Note $\dim(\ker T) + \dim(\operatorname{range} T) = 3 + 1$

$$= 4 = \dim(\mathrm{Mat}_{2\times 2})$$

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4. The standard basis for $\mathcal{P}_2$ is $\mathcal{E} = \{1, x, x^2\}$, and we have another basis, given by $\mathcal{B} = \{1, x-1, (x-1)^2\}$. Let $p(x) = 2 - x + 3x^2 \in \mathcal{P}_2$. Express $p(x)$ in terms of the basis $\mathcal{B}$ by following these steps:

   1. Write the change of basis matrix $P_{\mathcal{E}\leftarrow\mathcal{B}}$. First identify the standard basis of $\mathcal{P}_2$ with the standard basis of $\mathbb{R}^3$, $[1]_{\mathcal{E}} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $[x]_{\mathcal{E}} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $[x^2]_{\mathcal{E}} = \begin{pmatrix}0\\0\\1\end{pmatrix}$, then write each basis vector from $\mathcal{B}$ in these standard coordinates, and concatenate them to form a 3×3 matrix.
   2. Compute $\left(P_{\mathcal{E}\leftarrow\mathcal{B}} \mid [p(x)]_{\mathcal{E}}\right) \xrightarrow{RREF} \left(I_3 \mid [p(x)]_{\mathcal{B}}\right)$ or compute $(P_{\mathcal{E}\leftarrow\mathcal{B}})^{-1}[p(x)]_{\mathcal{E}}$ to find the coordinate vector $[p(x)]_{\mathcal{B}}$ in $\mathcal{B}$-coordinates.
   3. Use your answer to step 2 to rewrite $p(x)$ as a linear combination of the elements of $\mathcal{B}$.

   Remark: Observe that you have just computed the Taylor polynomial of degree 2 about $a = 1$ for $p(x)$, without taking any derivatives. Similarly, you could compute Taylor polynomials of degree n about any number a, using the basis $\{1, x-a, \ldots, (x-a)^n\}$.

1) $[1]_{\mathcal{E}} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $[x-1]_{\mathcal{E}} = [x]_{\mathcal{E}} - [1]_{\mathcal{E}} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$

$$[(x-1)^2]_{\mathcal{E}} = [x^2-2x+1]_{\mathcal{E}}$$
$$= [x^2]_{\mathcal{E}} - 2[x]_{\mathcal{E}} + [1]_{\mathcal{E}} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$$

so $P_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix}1 & -1 & 1\\0 & 1 & -2\\0 & 0 & 1\end{pmatrix}$

2) $\left(\begin{array}{ccc|ccc}1 & -1 & 1 & 1 & 0 & 0\\0 & 1 & -2 & 0 & 1 & 0\\0 & 0 & 1 & 0 & 0 & 1\end{array}\right) \xrightarrow{R_1+R_2 \mapsto R_1} \left(\begin{array}{ccc|ccc}1 & 0 & -1 & 1 & 1 & 0\\0 & 1 & -2 & 0 & 1 & 0\\0 & 0 & 1 & 0 & 0 & 1\end{array}\right) \xrightarrow[R_2+2R_3 \mapsto R_2]{R_1+R_3 \mapsto R_1}$

$\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 1 & 1\\0 & 1 & 0 & 0 & 1 & 2\\0 & 0 & 1 & 0 & 0 & 1\end{array}\right)$, $P_{\mathcal{B}\leftarrow\mathcal{E}} = \begin{pmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 1\end{pmatrix}$

$$[p(x)]_{\mathcal{E}} = [2-x+3x^2]_{\mathcal{E}} = 2[1]_{\mathcal{E}} - [x]_{\mathcal{E}} + 3[x^2]_{\mathcal{E}} = \begin{pmatrix}2\\-1\\3\end{pmatrix}$$

$$[p(x)]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{E}}[p(x)]_{\mathcal{E}} = \begin{pmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}2\\-1\\3\end{pmatrix} = \begin{pmatrix}2-1+3\\-1+6\\3\end{pmatrix} = \begin{pmatrix}4\\5\\3\end{pmatrix}$$

3) $4 + 5(x-1) + 3(x-1)^2 = p(x)$

5. Show that the linear transformation $T : \mathcal{P}_2 \to \mathcal{P}_2$ defined by $T(p(x)) = p(1-2x)$ is an isomorphism.

Since domain T = codomain T
and dim(domain T) = dim(ker T) + dim(range T)
and T is linear, it is enough to that
T is either 1-1 or onto, to show it is
both (i.e. an isomorphism). (b/c $\dim(P_2) < \infty$)

To show it is 1-1, we can show that ker T = 0.

Suppose $p(x) = a + bx + cx^2$ and

$$T(p(x)) = 0.$$

$$\begin{aligned} T(p(x)) &= p(1-2x) \\ &= a + b(1-2x) + c(1-2x)^2 \\ &= a + b - 2bx + c - 4cx + 4cx^2 \\ &= (a+b+c) + (-2b-4c)x + 4cx^2 \\ &= 0 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} a+b+c &= 0 \\ -2b-4c &= 0 \\ 4c &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right) \end{array}$$

$$\xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

So unique solution is
$a = b = c = 0$, which means
$\ker T = \{p(x) = 0\} = 0$