Name: Date: 02/29/2024

M20580 L.A. and D.E.
Worksheet 6

1. Let $T : \mathcal{P}^2 \to \mathcal{P}^2$ be the linear transformation on the space of degree two polynomials defined by

$$T(p(x)) = p(x+3)$$

Let $\mathcal{E} = \{1, x, x^2\}$ be the standard basis for $\mathcal{P}^2$. Let, $p(x)$ be the vector $2-3x+2x^2$ in $\mathcal{P}^2$.

(a) Apply the transformation T to the vector $p(x)$ to get $T(p(x))$.

(b) Write $p(x)$ in terms of the basis $\mathcal{E}$ to get $[p(x)]_{\mathcal{E}}$.

(c) Find the matrix of the linear transformation T with respect to the basis $\mathcal{E}$, $[T]_{\mathcal{E}\leftarrow\mathcal{E}}$.

(d) Compute $[T(p(x))]_{\mathcal{E}}$ by the matrix multiplication $[T]_{\mathcal{E}\leftarrow\mathcal{E}}[p(x)]_{\mathcal{E}}$. Use this to recover $T(p(x))$. Check your answer matches part (a).

Solution:

(a)

$$\begin{aligned} T(p(x)) =& (2-3(x+3)+2(x+3)^2) \\ =& 2-3x-9+2x^2+12x+18 \\ =& 11+9x+2x^2 \end{aligned}$$

(b) $[p(x)]_{\mathcal{E}} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$.

(c)

$$[T]_{\mathcal{E}\leftarrow\mathcal{E}} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & 9 \end{bmatrix}$$

(d)

$$[T]_{\mathcal{E}\leftarrow\mathcal{E}}[p(x)]_{\mathcal{E}} = \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \\ 2 \end{bmatrix}.$$

2. Compute the determinants using cofactor expansion along any row or column that seems convenient.

(a) $\begin{bmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 & 3 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{bmatrix}$

Solution:

(a) Using cofactor expansion across the first row we get

$$1((1)(3) - (1)(1)) - 0((5)(3) - (0)(1)) + 3((5)(1) - (0)(1)) = 2 + 0 + 15 = 17$$

(b) Using cofactor expansion across the second row we get

$$-2((1)(1) - (-1)(-2)) + 0((1)(1) - (-1)(-3)) - 1((1)(-2) - (3)(1)) = 2 + 0 + 5 = 7$$

(c) Using cofactor expansion down the second column we get

$$-(-1) \times \det\left(\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 2 \\ 2 & 1 & -3 \end{bmatrix}\right) =$$

$$2((2)(-3) - (1)(2)) - 3((1)(-3) - (2)(2)) + (-1)((1)(1) - (2)(2)) = -16 + 21 + 1 = 6$$

3. Use properties of determinants to evaluate the given determinant by inspection. Explain your reasoning.

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & -2 \\ 4 & 4 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution:

(a) The determinant is 0 as the third row is a multiple of the first row.

(b) The determinant of an upper triangular matrix is the product of the diagonal entries, in this case $(3)(-2)(4) = -24$

(c) Swapping two rows or columns has the effect of multiplying the determinant by -1. In this example, swaping the second and third column results in the identity matrix which has determinant 1, so the original matrix of part (c), has determinant -1.

4. Use Cramer's rule to solve the linear system

$$\begin{aligned} x + y - z &= 1 \\ x + y + z &= 2 \\ x - y \quad\; &= 3 \end{aligned}$$

Solution: The associated matrix is $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ which has determinant 4 and the associated solution vector is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ Replacing the first column with the solution vector we obtain the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{bmatrix}$ which has determinant 9. So $x = 9/4$.

Replacing the second column with the solution vector we obtain the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix}$ which has determinant -3. So $y = -3/4$.

Replacing the third column with the solution vector we obtain the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix}$ which has determinant 2. So $z = 2/4$.

Name: Date: 02/29/2024

5. Let A and B be $n \times n$ matrices with det $A = 3$ and det $B = -2$. Find the indicated determinants.

(a) $\det(AB)$

(b) $\det(A^2)$

(c) $\det(B^{-1}A)$

(d) $\det(2A)$

(e) $\det(3B^T)$

Solution:

(a) $\det(AB) = \det(A)\mathrm{B} = (3)(-2) = -6$

(b) $\det(A^2) = \det(AA) = \det(A)\det(A) = 3^2 = 9$

(c) $\det(B^{-1}A) = \det(B^{-1})\det(A) = \dfrac{1}{\det(B)}\det(A) = \dfrac{1}{-2}(3) = -3/2.$

(d) $\det(2A) = 2\det(A) = 6$

(e) $\det(3B^T) = 3\det(B^T) = 3\det(B) = 3(-2) = -6.$