Name: Date: 03/21/2024

Math 20580 L.A. and D.E. Tutorial
Worksheet 7

1. Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1).$$

   (a) What is the matrix of T with respect to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$.

   (b) If $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathcal{C} = \{\beta_1, \beta_2\}$, where

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1), \quad \alpha_3 = (1, 0, 0), \quad \beta_1 = (0, 1), \quad \beta_2 = (1, 0),$$

   what is the matrix of $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$?

Solution:

(a) Since $T(1, 0, 0) = (1, -1)$, $T(0, 1, 0) = (1, 0)$, $T(0, 0, 1) = (0, 2)$, then the matrix of T is $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$.

(b) First, $T(1, 0, -1) = (1, -3) = -3 \cdot (0, 1) + 1 \cdot (1, 0)$. Next, $T(1, 1, 1) = (2, 1)$ $= 1 \cdot (0, 1) + 2 \cdot (1, 0)$. Lastly, $T(1, 0, 0) = (1, -1) = -1(0, 1) + 1(1, 0)$. So, $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$ is $\begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

2. Determine which of the following are true or false, and justify your answer. Let A and B both be $n \times n$ matrices.

   (a) If A and B are similar, then they have the same eigenvalues.

   (b) If A and B are similar, then they have the same eigenvectors.

   (c) If A and B have the same characteristic polynomial, then they are similar.

   (d) If $\vec{v}$ is an eigenvector for both A and B, then it is an eigenvector for AB.

   (e) If A has 0 as an eigenvalue, then $\det A = 0$.

   (f) If A is invertible then it is diagonalizable.

Solution:

(a) **True**: Let $B = PAP^{-1}$. Then

$$B - \lambda I = PAP^{-1} - \lambda I = PAP^{-1} - P(\lambda I)P^{-1} = P(A - \lambda I)P^{-1}$$

Finding the determinant of the left and right side gives:

$$\begin{aligned} \det(B - \lambda I) &= \det(P)\det(A - \lambda I)\det(P^{-1}) \\ &= \det(P)\det(P^{-1})\det(A - \lambda I) \\ &= \det(A - \lambda I) \end{aligned}$$

So A and B have the same characteristic polynomial, hence the same eigenvalues.

(b) **False**: Let $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, and let $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then D and $PDP^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ are similar, and in particular have the same eigenvalues. However, D has $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as an eigenvector, while PDP^{-1} does not.

(c) **False**: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ have the same characteristic polynomial, $(\lambda - 1)^2$. However, they are not similar: if they were similar they would have to have the same geometric multiplicity for $\lambda = 1$, but 1 has geometric multiplicity 2 for A and 1 for B.

(d) **True**: Let v be a λ-eigenvector for A, and a λ'-eigenvector for B. Then $(AB)v = A(Bv) = A(\lambda' v) = \lambda'(Av) = \lambda'\lambda v$, so v is a $\lambda'\lambda$-eigenvector for AB.

(e) **True**: Let v be a nonzero 0-eigenvalue for A. Then $Av = 0v = 0$, so the system $Ax = \vec{0}$ has a nontrivial solution, so A is not invertible, so $\det A = 0$.

(f) **False**: Consider the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. There is only one eigenvalue $\lambda = 1$, with algebraic multiplicity 2 and geometric multiplicity 1. Since the geometric multiplicity is less than the algebraic multiplicity, this matrix is not diagonalizable.

Name: Date: 03/21/2024

3. Let $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

   (a) Find an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

   (b) What is A^{2024}?

Solution:

(a) • First, we find all eigenvalues of A:

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix} = \lambda^2 - 3\lambda - 10 = (\lambda+2)(\lambda-5),$$

which implies that the eigenvalues are $\lambda = -2$ and $\lambda = 5$. Since the eigenvalues of A are **distinct**, A is **diagonalizable**.

• Next, we find eigenvectors corresponding to each eigenvalue:

- For $\lambda = -2$:

$$(A + 2I_2)\mathbf{x} = 0 \Longleftrightarrow \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,$$

and hence

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 4 \\ -3 \end{bmatrix},\ t \in \mathbb{R}.$$

- For $\lambda = 5$: Using the same method, the eigenvectors are

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ t \in \mathbb{R}.$$

• According to the theory of diagonalization, the following matrices

$$P = \begin{bmatrix} 4 & 1 \\ -3 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

satisfies $A = PDP^{-1}$.

(b) Using that $A = PDP^{-1}$, we can simplify:

$$\begin{aligned} A^{2024} &= (PDP^{-1})^{2024} = (PDP^{-1})(PDP^{-1})\dots(PDP^{-1}) \qquad \text{(2024 times)} \\ &= PD(P^{-1}P)D(P^{-1}P)D\dots(P^{-1}P)DP^{-1} \\ &= PD^{2024}P^{-1} \end{aligned}$$

Let's compute P^{-1} and D^{2024}:

$$P = \begin{bmatrix} 4 & 1 \\ -3 & 1 \end{bmatrix} \Longrightarrow P^{-1} = \frac{1}{7}\begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix};$$

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} \Longrightarrow D^{2024} = \begin{bmatrix} (-2)^{2024} & 0 \\ 0 & 5^{2024} \end{bmatrix}.$$

Finally,

$$\begin{aligned} A^{2024} &= PD^{2024}P^{-1} \\ &= \begin{bmatrix} 4 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} (-2)^{2024} & 0 \\ 0 & 5^{2024} \end{bmatrix} \frac{1}{7}\begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} (3)5^{100} + (4)2^{2024} & 4 \cdot 5^{2024} - (4)2^{2024} \\ (3)5^{2024} - (3)2^{2024} & 3 \cdot 2^{2024} + (4)5^{2024} \end{bmatrix} \end{aligned}$$

Name: Date: 03/21/2024

4. Let $A = \begin{bmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{bmatrix}$.

(a) Determine all the eigenvalues of A.

(b) For each eigenvalue λ of A, find the eigenspace E_λ.

(c) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

(d) Determine an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution:

(a) We have

$$\det(A - \lambda I_3) = -(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

The eigenvalues are $\lambda_1 = 1, \lambda_2 = 2$, and $\lambda_3 = 3$.

(b) Recall that a λ-eigenvector is an element of the kernel of $A - \lambda I$.

We have that

$$A-\lambda_1 I = \begin{bmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{bmatrix} \quad A-\lambda_2 I = \begin{bmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{bmatrix} \quad A-\lambda_3 I = \begin{bmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{bmatrix}$$

The eigenspaces E_λ of A will be the null spaces $A - \lambda I$. We find these using row reduction on the homogeneous systems $[A - \lambda I | \vec{0}]$.

For $\lambda_1 = 1$, we have $E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$.

For $\lambda_2 = 2$, we have $E_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda_3 = 3$, we have $E_3 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

(c) A basis for $\mathbb{R}^3$ consisting of eigenvectors of A is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

(d) The matrices P and D we need to find are

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Name: Date: 03/21/2024

5. Eigenvalues and eigenvectors only make sense for square matrices. Fortunately, even if A is a nonsquare matrix, AA^T and A^TA will be square. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}$.

   (a) Find the characteristic polynomial and eigenvalues of AA^T.

   (b) Find the characteristic polynomial and eigenvalues of A^TA.

   (c) What do you notice about the eigenvalues of the two square matrices?

Solution:

(a) $AA^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -4 \\ 0 & -4 & 8 \end{bmatrix}$. Using Laplace expansion along the top row, we find that the characteristic polynomial of AA^T is $\det(AA^T - \lambda I) = (2-\lambda)((2-\lambda)(8-\lambda) - 16) = (2-\lambda)(\lambda^2 - 10\lambda) = (2-\lambda)(\lambda)(\lambda - 10)$. The eigenvalues of AA^T are the roots of the characteristic polynomial, namely $\lambda = 0, 2, 10$.

(b) $A^TA = \begin{bmatrix} 6 & -4 \\ -4 & 6 \end{bmatrix}$. The characteristic polynomial of A^TA is $\det(A^TA - \lambda I) = (6-\lambda)(6-\lambda) - 16 = \lambda^2 - 12\lambda + 20 = (\lambda - 2)(\lambda - 10)$. The eigenvalues of A^TA are the roots of the characteristic polynomial, namely $\lambda = 2, 10$.

(c) Each eigenvalue of A^TA is also an eigenvalue of AA^T, but not all eigenvalues of AA^T is an eigenvalue of A^TA.