Name: SOLUTIONS Date: 03/28/2024

M20580 L.A. and D.E. Tutorial
Worksheet 8

1. Find all eigenvalues and a eigenvector for each eigenvalue of

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

Solution: The characteristic equation is

$$\det(A - \lambda I) = \lambda^2 - 2\lambda + 5.$$

Then the eigenvalues are

$$\lambda = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i.$$

The REF of $A - (1+2i)I$ is

$$\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus the eigenvalue $\mathbf{v}_1$ associated with $\lambda_1 = 1 + 2i$ is $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and the eigenvalue $\mathbf{v}_2$ associated with $\lambda_2 = 1 - 2i = \overline{\lambda_1}$ is $\begin{bmatrix} -i \\ 1 \end{bmatrix} = \overline{\mathbf{v}}_1$.

2. Determine if the given vectors form an orthogonal set

$$\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Solution: Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the given vectors respectively. Then we have

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= -3 + 2 + 1 = 0 \\ \mathbf{v}_1 \cdot \mathbf{v}_3 &= -3/2 - 2 + 7/2 = 0 \\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= 1/2 - 4 + 7/2 = 0. \end{aligned}$$

Then they form an orthogonal set.

3. Determine if the given vectors form an orthogonal basis for $\mathbb{R}^2$

$$\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}-6\\9\end{bmatrix}.$$

Solution:

(1) Proving directly: Clearly, they form an orthogonal set. We only need to check if they are linearly independent. That is

$$\begin{bmatrix}3 & -6\\2 & 9\end{bmatrix} \sim \begin{bmatrix}1 & -2\\2 & 9\end{bmatrix} \sim \begin{bmatrix}1 & -2\\0 & 13\end{bmatrix}.$$

(2) Using Theorem 5.1 in Poole: *If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is an orthogonal set of nonzero vector in $\mathbb{R}^n$, then these vectors are linearly independent.* Since our vectors are nonzero and form an orthogonal set, they are also linearly independent. Thus they form an orthogonal basis for $\mathbb{R}^2$.

4. Find the orthogonal complement $W^{\perp}$ of W in $\mathbb{R}^3$ where

$$W = \left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} : 2x + y = z\,, y + z = 0 \right\}.$$

Hint: $(\text{null}A)^{\perp} = \text{col}A^T$ and $(\text{col}A)^{\perp} = \text{null}A^T$.

Solution: Note that $W = \text{null}\{\begin{bmatrix}2 & 1 & -1\\0 & 1 & 1\end{bmatrix}\}$. Let $A = \begin{bmatrix}2 & 1 & -1\\0 & 1 & 1\end{bmatrix}$. Then $W^{\perp} = \text{col}A^T$. We have that

$$A^T = \begin{bmatrix}2 & 0\\1 & 1\\-1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}.$$

Thus $W^{\perp} = \text{span}\left\{ \begin{bmatrix}2\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$.

5. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) Find $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$

Solution: $\mathbf{u} \cdot \mathbf{v} = 1 - 2 + 0 = -1 = \mathbf{v} \cdot \mathbf{u}$.

(b) Find $\text{proj}_{\mathbf{u}}\mathbf{v}$ and $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Solution:

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{-1}{2}\mathbf{u}$$
$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{-1}{14}\mathbf{v}.$$

6. Find the orthogonal projection of $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ onto the subspace $W = \text{span}(\mathbf{u}_1, \mathbf{u}_2)$ in $\mathbb{R}^3$, where $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. Find the distance from $\mathbf{v}$ to W.

Solution: Note that $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, then $\mathbf{u}_1$ and $\mathbf{u}_2$ form an orthogonal basis of W. Thus,

$$\text{proj}_W(\mathbf{v}) = \frac{(1,2,-3) \cdot (1,0,1)}{\|(1,0,1)\|^2}(1{,}0,1) + \frac{(1,2,-3) \cdot (-1,2,1)}{\|(-1,2,1)\|^2}(-1,2,1)$$
$$\text{proj}_W(\mathbf{v}) = (-1,0,-1)$$
$$\text{perp}_W(\mathbf{v}) = v - \text{proj}_W(\mathbf{v}) = (2,2,-2).$$

Then distance from $\mathbf{v}$ to W is $\|\mathbf{v} - \text{proj}_W(\mathbf{v})\| = 2\sqrt{3}$.

7. Apply the Gram-Schmidt process to the vectors $(0,1,1),(-2,0,-2)$ and $(-2,0,-1)$ to obtain an orthonormal basis.

Solution: Let $\mathbf{v}_1 = (0,1,1)$. Then

$$\mathbf{v}_2 = (-2,0,-2) - \frac{(-2,0,-2)\cdot(0,1,1)}{\|(0,1,1)\|^2}(0,1,1) = (-2,1,-1),$$

and

$$\begin{aligned}\mathbf{v}_3 &= (-2,0,-1) - \left(\frac{(-2,0,-1)\cdot(0,1,1)}{\|(0,1,1)\|^2}(0,1,1) + \frac{(-2,0,-1)\cdot(-2,1,-1)}{\|(-2,1,-1)\|^2}(-2,1,-1)\right)\\ &= (-1/3,-1/3,1/3)\end{aligned}$$

To get orthonormal basis, we simply normalize the above vector, i.e.

$$\begin{aligned}\mathbf{u}_1 &= \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = (0,1/\sqrt{2},1/\sqrt{2})\\ \mathbf{u}_2 &= \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = (-2/\sqrt{6},1/\sqrt{6},-1/\sqrt{6})\\ \mathbf{u}_3 &= \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = (-1/\sqrt{3},-1\sqrt{3},1\sqrt{3}).\end{aligned}$$