Name: Date: 04/04/2024

M20580 L.A. and D.E. Tutorial
Worksheet 9

1. Let W be a subspace of $\mathbb{R}^3$ such that

$$W = \text{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}3\\4\\2\end{bmatrix}\right\}$$

   (a) Use the Gram-Schmidt process to obtain an orthogonal basis of W.

   (b) Find the orthogonal decomposition of $\mathbf{v} = \begin{bmatrix}4\\-4\\3\end{bmatrix}$ with respect to W.

(a) $\vec{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$

$$\vec{v}_2 = \begin{bmatrix}3\\4\\2\end{bmatrix} - \frac{7}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\2\end{bmatrix}$$

$$\Rightarrow W = \text{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1/2\\1/2\\2\end{bmatrix}\right\}$$

(b) $$\text{proj}_W\vec{v} = \frac{\vec{v}\cdot\vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\vec{v}\cdot\vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= 0\cdot\vec{v}_1 + \frac{4}{9}v_2 = \begin{bmatrix}-2/9\\2/9\\8/9\end{bmatrix}$$

$$\text{perp}_W\vec{v} = \vec{v} - \text{proj}_W\vec{v} = \begin{bmatrix}38/9\\-38/9\\19/9\end{bmatrix}$$

$$\Rightarrow \vec{v} = \begin{bmatrix}-2/9\\2/9\\8/9\end{bmatrix} + \begin{bmatrix}38/9\\-38/9\\19/9\end{bmatrix}$$

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2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

We want to find the QR factorization of A.

(a) Use the Gram-Schmidt process to transform the columns of A into an orthogonal basis.

(b) Transform the orthogonal basis obtained in part (a) into an orthonormal basis by dividing each vector by its magnitude.

(c) The orthonormal vectors obtained in part (b) will now be the columns of your matrix Q (remember that order of vectors matters!)

(d) Use the fact that $Q^T A = R$ to find your R matrix.

(a) $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$

(b) $\hat{v}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\hat{v}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(c) $Q = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(d) $R = Q^T A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

3. Find a least squares solution to $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Furthermore, is the solution unique or not? Justify your answer.

$$A^TA = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ 6 & 6 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Note that the columns of A^TA are not multiples of each other and so are linearly independent. Therefore, A^TA is invertible with inverse:

$$(A^TA)^{-1} = \frac{1}{30} \begin{bmatrix} 6 & -6 \\ -6 & 11 \end{bmatrix}$$

$$\Rightarrow \vec{x} = (A^TA)^{-1}A^T\vec{b} = \frac{1}{30} \begin{bmatrix} 6 & -6 \\ -6 & 11 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 7/15 \end{bmatrix}$$

Furthermore, because A^TA is invertible the solution is unique.

4. Consider the following data points:

$$(0,4)\ ,\ (1,1)\ ,\ (2,0)$$

We want to find the least squares approximating line.

(a) We transform these data points into a system of 3 linear equations $y_i = a + bx_i$ and construct our matrix A and vector $\mathbf{b}$ from this information.

(b) We then find the least squares solution to $A\mathbf{x} = \mathbf{b}$.

(c) For a least squares solution $\overline{\mathbf{x}} = \begin{bmatrix} a \\ b \end{bmatrix}$, the least squares approximating line is $y = a+bx$.

(a) Using the data points, we get the following system of equations:

$$\left.\begin{aligned} a+b\cdot 0 &= 4 \\ a+b\cdot 1 &= 1 \\ a+b\cdot 2 &= 0 \end{aligned}\right\} \Rightarrow A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix},\ \vec{b} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}.$$

(b) $$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\Rightarrow (A^TA)^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 11/3 \\ -2 \end{bmatrix}$$

(c) $y = 11/3 - 2x$

5. Consider the following data points:

$$(1,1)\ ,\ (2,-2)\ ,\ (3,3)\ ,\ (4,4)$$

Find the least squares approximating parabola for the given points (Hint: the program is almost exactly the steps laid out in problem 4, except our linear equations will take the form $y_i = a + bx_i + cx_i^2$ and similarly for the final answer).

(a) Same as #4, we get a system of equations

$$\left.\begin{aligned} a+b\cdot 1+c\cdot 1^2 &= 1\\ a+b\cdot 2+c\cdot 2^2 &= -2\\ a+b\cdot 3+c\cdot 3^2 &= 3\\ a+b\cdot 4+c\cdot 4^2 &= 4 \end{aligned}\right\} \Rightarrow A = \begin{bmatrix} 1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 3 & 9\\ 1 & 4 & 16 \end{bmatrix},\ \vec{b} = \begin{bmatrix} 1\\ -2\\ 3\\ 4 \end{bmatrix}$$

(b)

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & 2 & 3 & 4\\ 1 & 4 & 9 & 16 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 3 & 9\\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 10 & 30\\ 10 & 30 & 100\\ 30 & 100 & 354 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & 2 & 3 & 4\\ 1 & 4 & 9 & 16 \end{bmatrix}\begin{bmatrix} 1\\ -2\\ 3\\ 4 \end{bmatrix} = \begin{bmatrix} 6\\ 22\\ 84 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 4 & 10 & 30 & 6\\ 10 & 30 & 100 & 22\\ 30 & 100 & 354 & 84 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_2]{R_2 \to \frac{1}{10}R_2} \left[\begin{array}{ccc|c} 1 & 3 & 10 & 11/5\\ 4 & 10 & 30 & 6\\ 30 & 100 & 354 & 84 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 30R_1]{R_2 \to R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & 3 & 10 & 11/5\\ 0 & -2 & -10 & -14/5\\ 0 & 10 & 54 & 18 \end{array}\right]$$

next page →

$$\left[\begin{array}{ccc|c} 1 & 3 & 10 & 11/5 \\ 0 & -2 & -10 & -14/5 \\ 0 & 10 & 54 & 18 \end{array}\right] \xrightarrow{R_3 \to R_3 + 5R_2} \left[\begin{array}{ccc|c} 1 & 3 & 10 & 11/5 \\ 0 & -2 & -10 & -14/5 \\ 0 & 0 & 4 & 4 \end{array}\right]$$

Using back substitution we get:

$$c = 1 \Rightarrow b = \frac{7}{5} - 5 = -\frac{18}{5} \Rightarrow a = 11/5 + 3 \cdot \frac{18}{5} - 10 = 3$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 3 \\ -18/5 \\ 1 \end{bmatrix}$$

(c) $y = 3 - \frac{18}{5}x + x^2$