Name: ______________________________

Section #: ______________________________

Exam 1, Math 20580
February 13, 2025

- The Honor Code *is* in affect for this examination. All work is to be your own.
- Please turn off and put away all cellphones and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for one hour and 15 minutes.
- Be sure that your name and your instructor's name and your section number are on the front page of your exam.

01	Matthew Dyer	8:20-9:10	127 HH
02	Matthew Dyer	9:25-10:15	127 HH
03	Jeffrey Diller	10:30-11:20	136 DBRT
04	Han Lu	11:30-12:20	114 PASQ
05	Sudipta Gosh	12:50-1:40	140 DBRT
06	Sudipta Gosh	2:00-2:50	140 DBRT
07	Claudia Polini	3:30-4:20	140 DBRT

- There are 12 problems: 8 are multiple choice and 4 are partial credit.
- Be sure that you have all 8 pages of this exam.
- Each multiple choice question is worth 6 points. Your score will be the sum of the best 7 scores on the multiple choice plus your score on questions 9–12.
- Make sure you have correctly marked your multiple choice answers by the time the exam ends. No credit will be given for incorrectly marked answers.

Please mark you answers with an **X**, not a circle.

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
			1		
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
			2		
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
			3		
7	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)
			4		

MC Total.	______
9.	______
10.	______
11.	______
12.	______
Total.	______

Multiple choice problems

1. (6 points) Which matrix below is equal to is the product AB, where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \ ?$$

(a) $\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

2. (6 points) Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Which of the following matrices is the reduced row echelon form (RREF) of A?

(a) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

3. (6 points) For which of the following matrices A is AA^T invertible (where A^T is the transpose of A)?

(I) $\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ (II) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (III) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (IV) $\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

(a) (III) only

(b) (III) and (IV) only

(c) (I), (II) and (III) only

(d) (I) and (III) only

(e) (I), (III) and (IV) only

4. (6 points) For which values of h and k is the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & h & 0 & 1 \\ 0 & 0 & k & 1 \end{bmatrix}$$

in reduced row echelon form (RREF)?

(a) $h = 1$ and any k

(b) no values of h and k are possible

(c) $(h, k) = (1, 0)$ and $(h, k) = (1, 1)$ only

(d) $(h, k) = (0, 0)$ and $(h, k) = (0, 1)$ only.

(e) $h = 0$ and any k

5. (6 points) Let A be a 4×5 matrix such that the dimension of the null space of the transpose matrix A^T is 1. What is the rank of A?

(a) 1 (b) 2 (c) 4 (d) 3 (e) 0

6. (6 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by counterclockwise rotation of the plane about the origin by angle π (in radians). Which of the following is the standard matrix of T?

(a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

7. (6 points) Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

in $\mathbb{R}^3$. Which of the following vectors is NOT in Span($\{\vec{v}_1, \vec{v}_2\}$)?

(a) $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$ (e) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

8. (6 points) Consider the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

for $\mathbb{R}^3$. Let

$$\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

be the vector in $\mathbb{R}^3$ such that the coordinate vector of $\vec{x}$ with respect to $\mathcal{B}$ is

$$[x]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}.$$

What is the value of a?

(a) 2 (b) 0 (c) -3 (d) 6 (e) -1

Partial credit problems

9. (18 points) The matrix A has a row echelon form (REF) B, as shown below:

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & -2 & 5 & -4 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 5 & -4 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Compute the rank of A.

(b) Compute the nullity of A.

(c) Find a basis of the column space $\mathrm{col}(A)$ of A. (Write your answer as a list of column vectors.)

(d) Find a basis of the row space $\mathrm{row}(A)$ of A. (Write your answer as a list of row vectors.)

(e) Find a basis of the null space $\mathrm{null}(A)$ of A. (Write your answer as a list of column vectors.)

10. (10 points) Find the inverse of the matrix below (if it exists). If the inverse does not exist, explain why.

$$\begin{bmatrix} -2 & 1 & 2 \\ 2 & -1 & 0 \\ -2 & 1 & 3 \end{bmatrix}.$$

11. (15 points)

(a) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_1 \end{bmatrix}.$$

Write down the standard matrix $A = [T]$ of T.

(b) Let $S = T \circ T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the composite of T with itself. Compute the standard matrix $[S] = [T \circ T]$ of S.

(c) Fill in the $\ldots$ below to give a formula for S like that given for T in (a):

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{bmatrix}$$

12. (15 points) Consider the two bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^2$ given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}.$$

(a) Let $\vec{x}$ be the vector in $\mathbb{R}^2$ with $\mathcal{B}$-coordinate vector

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

Find $\vec{x}$ explicitly as a vector in $\mathbb{R}^2$ i.e. what are the values of a and b such that $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$.

(b) Find the change of coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{C}$ (recall that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the matrix such that $[\vec{v}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot [\vec{v}]_{\mathcal{B}}$ for all vectors $\vec{v}$ in $\mathbb{R}^2$).

(c) Find the $\mathcal{C}$-coordinate vector $[\vec{x}]_{\mathcal{C}}$ of the vector $\vec{x}$ in (a).