Name: ______________________

Section #: ______________________

Exam 1, Math 20580
February 13, 2025

- The Honor Code *is* in affect for this examination. All work is to be your own.
- Please turn off and put away all cellphones and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for one hour and 15 minutes.
- Be sure that your name and your instructor's name and your section number are on the front page of your exam.
- There are 12 problems: 8 are multiple choice and 4 are partial credit.
- Be sure that you have all 8 pages of this exam.
- Each multiple choice question is worth 6 points. Your score will be the sum of the best 7 scores on the multiple choice plus your score on questions 9–12.
- Make sure you have correctly marked your multiple choice answers by the time the exam ends. No credit will be given for incorrectly marked answers.

01	Matthew Dyer	8:20-9:10	127 HH
02	Matthew Dyer	9:25-10:15	127 HH
03	Jeffrey Diller	10:30-11:20	136 DBRT
04	Han Lu	11:30-12:20	114 PASQ
05	Sudipta Gosh	12:50-1:40	140 DBRT
06	Sudipta Gosh	2:00-2:50	140 DBRT
07	Claudia Polini	3:30-4:20	140 DBRT

Please mark you answers with an **X**, not a circle.

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
			1		
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
			2		
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
			3		
7	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)
			4		

MC Total.	
9.	
10.	
11.	
12.	
Total.	

Multiple choice problems

1. (6 points) Which matrix below is equal to is the product AB, where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \text{ ?}$$

(a) $\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 0\cdot 0 + -1\cdot 0 & 1\cdot 1 + 0\cdot 1 + (-1)\cdot(-1) \\ (-1)\cdot 1 + 1\cdot 0 + 0\cdot 0 & (-1)\cdot 1 + 1\cdot 1 + 0\cdot(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

2. (6 points) Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Which of the following matrices is the reduced row echelon form (RREF) of A?

(a) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

A is already in RREF.

3. (6 points) For which of the following matrices A is AA^T invertible (where A^T is the transpose of A)?

$$\text{(I)} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(II)} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{(III)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{(IV)} \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

(a) (III) only

(b) (III) and (IV) only

(c) (I), (II) and (III) only

(d) (I) and (III) only

(e) (I), (III) and (IV) only

The ranks of A in (I)–(IV) are 2, 1, 2, 3.
One has $\operatorname{rank}(AA^T) = \operatorname{rank}(A)$ and AA^T is invertible if its rank is equal to its size (number of rows and columns).
So I, III, IV only are invertible.
Another way A is invertible in III, IV (square with linearly independent columns), so AA^T is invertible in III, IV with $(AA^T)^{-1} = (A^{-1})^T A^{-1}$. In I, $AA^T = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ is invertible since its determinant is $2 \cdot 1 - (-1)\cdot(-1) = 1 \neq 0$. In II, $AA^T = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ is not invertible since its determinant is $(-1)(-1) - 1 \cdot 1 = 0$.

4. (6 points) For which values of h and k is the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & h & 0 & 1 \\ 0 & 0 & k & 1 \end{bmatrix}$$

in reduced row echelon form (RREF)?

(a) $h = 1$ and any k

(b) no values of h and k are possible

(c) $(h, k) = (1, 0)$ and $(h, k) = (1, 1)$ only

(d) $(h, k) = (0, 0)$ and $(h, k) = (0, 1)$ only.

(e) $h = 0$ and any k

h can't be a leading entry since there is a non-zero entry above it (1). So if A is RREF, then $h = 0$, and 1 is the leading entry of the second row. But then there is a non-zero entry below the pivot 1 in the second row, and A is not in RREF.

5. (6 points) Let A be a 4×5 matrix such that the dimension of the null space of the transpose matrix A^T is 1. What is the rank of A?

(a) 1 (b) 2 (c) 4 (d) 3 (e) 0

A^T is a 5×4 matrix, nullity$(A^T) = 1$.
rank(A^T) + nullity(A^T) = number of columns of $A^T = 4$,
so rank$(A^T) = 3$. Finally, rank(A) = rank$(A^T) = 3$.

6. (6 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by counterclockwise rotation of the plane about the origin by angle π (in radians). Which of the following is the standard matrix of T?

(a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $T(\vec{e_1}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $T(\vec{e_2}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

So $[T] = [T(\vec{e_1}) \; T(\vec{e_2})] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

7. (6 points) Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

in $\mathbb{R}^3$. Which of the following vectors is NOT in Span($\{\vec{v}_1, \vec{v}_2\}$)?

(a) $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$ (e) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$a\vec{v}_1 + b\vec{v}_2 = \begin{bmatrix} 3a+3b \\ b \\ a \end{bmatrix}$. Taking $(a,b) = (2,-1)$, $(1,-1)$, $(-2,2)$, $(0,0)$ gives the vectors in (a), (b), (d), (e). To get (c), we would need $a=-2, b=1$ but then $a\vec{v}_1 + b\vec{v}_2 = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \neq \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

8. (6 points) Consider the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

for $\mathbb{R}^3$. Let

$$\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

be the vector in $\mathbb{R}^3$ such that the coordinate vector of $\vec{x}$ with respect to $\mathcal{B}$ is

$$[x]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}.$$

What is the value of a?

(a) 2 (b) 0 (c) -3 (d) 6 (e) -1

$\vec{x} = 2\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$ so $a=6$.

Partial credit problems

9. (18 points) The matrix A has a row echelon form (REF) B, as shown below:

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & -2 & 5 & -4 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 5 & -4 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Compute the rank of A. rank(A) = #(pivots of B) $= 4$.

(b) Compute the nullity of A. nullity (A) = #(columns of A) − (rank A) $= 5 - 4 = 1$.

(c) Find a basis of the column space col(A) of A. (Write your answer as a list of column vectors.)

Take the columns of A corresponding to a column of B with a pivot:

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

(d) Find a basis of the row space row(A) of A. (Write your answer as a list of row vectors.)

Take the rows of B containing a pivot of B

$[1\ 0\ -1\ 1\ 0]$, $[0\ 1\ 0\ 1\ 0]$ $[0\ 0\ -1\ 5\ -4]$ $[0\ 0\ 0\ 4\ -4]$

(e) Find a basis of the null space null(A) of A. (Write your answer as a list of column vectors.)

Solve $\begin{cases} x_1 - x_3 + x_4 = 0 \\ x_2 + x_4 = 0 \\ -x_3 + 5x_4 - 4x_5 = 0 \\ 4x_4 - 4x_5 = 0 \end{cases}$ by back substitution: $x_4 = x_5$, $x_3 = 5x_4 - 4x_5 = x_5$, $x_2 = -x_4 = -x_5$, $x_1 = x_3 - x_4 = 0$.

Null space $= \left\{ \begin{bmatrix} 0 \\ -x_5 \\ x_5 \\ x_5 \\ x_5 \end{bmatrix} \right\} = \left\{ x_5 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, basis $= \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Another way Reduce B to RREF: $B \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Solve $\begin{cases} x_1 = 0 \\ x_2 + x_5 = 0 \\ x_3 - x_5 = 0 \\ x_4 - x_5 = 0 \end{cases}$ to get $\begin{cases} x_1 = 0 \\ x_2 = -x_5 \\ x_3 = x_5 \\ x_4 = x_5 \\ x_5 = x_5 \end{cases}$

The null space and its basis are then as in the first method.

10. (10 points) Find the inverse of the matrix below (if it exists). If the inverse does not exist, explain why.

$$A = \begin{bmatrix} -2 & 1 & 2 \\ 2 & -1 & 0 \\ -2 & 1 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} -2 & 1 & 2 \\ 2 & -1 & 0 \\ -2 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$ which is in REF.

So A has rank 2, its RREF is not I_3. So <u>A is not invertible</u> by the invertible matrix theorem, or by noting that if A was invertible, $[A \mid I_3]$ would be row equivalent to $[I_3 \mid A^{-1}]$ by the Gauss-Jordan method for computing A^{-1}, and so A would be row equivalent to I_3.

11. (15 points)

(a) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_1 \end{bmatrix}.$$

Write down the standard matrix $A = [T]$ of T.

$$T(\vec{e_1}) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad T(\vec{e_2}) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad T(\vec{e_3}) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

$$[T] = [T(\vec{e_1})\ T(\vec{e_2})\ T(\vec{e_3})] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

(b) Let $S = T \circ T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the composite of T with itself. Compute the standard matrix $[S] = [T \circ T]$ of S.

$$[S] = [T \circ T] = [T][T] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}$$

(c) Fill in the ... below to give a formula for S like that given for T in (a):

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 + x_3 \\ x_2 - 2x_3 + x_1 \\ x_3 - 2x_1 + x_2 \end{bmatrix}$$

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + x_3 \\ x_1 + x_2 - 2x_3 \\ -2x_1 + x_2 + x_3 \end{bmatrix}$$

12. (15 points) Consider the two bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^2$ given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}.$$

(a) Let $\vec{x}$ be the vector in $\mathbb{R}^2$ with $\mathcal{B}$-coordinate vector

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

Find $\vec{x}$ explicitly as a vector in $\mathbb{R}^2$ i.e. what are the values of a and b such that $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$.

$$\vec{x} = 4\begin{bmatrix} 1 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \end{bmatrix} \qquad (a=2,\ b=-7).$$

(b) Find the change of coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{C}$ (recall that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the matrix such that $[\vec{v}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot [\vec{v}]_{\mathcal{B}}$ for all vectors $\vec{v}$ in $\mathbb{R}^2$).

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 3 & 5 & -2 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 0 & -1 & -5 & -7 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 0 & 1 & 5 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -9 & -12 \\ 0 & 1 & 5 & 7 \end{array}\right]$$

$$= \left[I_2 \,\middle|\, \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \right].$$

$$\text{So } \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} -9 & -12 \\ 5 & 7 \end{bmatrix}$$

(c) Find the $\mathcal{C}$-coordinate vector $[\vec{x}]_{\mathcal{C}}$ of the vector $\vec{x}$ in (a).

$$[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -9 & -12 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -24 \\ 13 \end{bmatrix}$$

Check: $-24\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 13\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \end{bmatrix} = \vec{x}$ (from (a)).