Name: ______________________________

Section #: ______________________________

Exam 2, Math 20580
March 6, 2025

- The Honor Code *is* in affect for this examination. All work is to be your own.
- Please turn off and put away all cellphones and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for one hour and 15 minutes.
- Be sure that your name and your instructor's name and your section number are on the front page of your exam.
- There are 12 problems: 8 are multiple choice and 4 are partial credit.
- Be sure that you have all 8 pages of this exam.
- Each multiple choice question is worth 6 points. Your score will be the sum of the best 7 scores on the multiple choice plus your score on questions 9–12.
- Make sure you have correctly marked your multiple choice answers by the time the exam ends. No credit will be given for incorrectly marked answers.

01	Matthew Dyer	8:20-9:10	127 HH
02	Matthew Dyer	9:25-10:15	127 HH
03	Jeffrey Diller	10:30-11:20	136 DBRT
04	Han Lu	11:30-12:20	114 PASQ
05	Sudipta Gosh	12:50-1:40	140 DBRT
06	Sudipta Gosh	2:00-2:50	140 DBRT
07	Claudia Polini	3:30-4:20	140 DBRT

Please mark you answers with an **X**, not a circle.

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
			1		
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
			2		
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
			3		
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)
			4		

MC Total.	________
9.	________
10	________
11.	________
12.	________
Total.	________

1. (6 points) Which of the following is a vector space?

I. $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x, y, z \text{ in } \mathbb{R} \text{ and } x + y + z = 1 \right\}$.

II. $\left\{ \begin{bmatrix} 0 \\ x \\ y \end{bmatrix} \middle| x, y \text{ in } \mathbb{R} \text{ and } y = x \text{ or } y = -x \right\}$.

III. $\left\{ \begin{bmatrix} t \\ t+s \\ s-t \end{bmatrix} \middle| s, t \text{ in } \mathbb{R} \right\}$.

IV. $\left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} \middle| s, t \text{ in } \mathbb{R} \text{ and } t \geq 0 \right\}$.

(a) III and IV. (b) IV only. (c) II, III and IV. (d) I and III. (e) III only.

✗ I Doesn't contain $\vec{0}$

✗ II Contains $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ but not $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

✓ III $\begin{bmatrix} t \\ t+s \\ s-t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. It is span$\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$

✗ IV Contains $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ but not $(-1)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

2. (6 points) Let $M_{2,2}$ denote the set of all 2×2 matrices, and let S be the subspace of $M_{2,2}$ consisting of all matrices A satisfying $A + A^T = 0$. What is the dimension of S?

(a) 2 (b) 3 (c) 1 (d) 0 (e) 4

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A + A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

when $a = d = 0$ and $b + c = 0$ ie $c = -b$.

S consists of matrices $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$, $S = \text{span}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is a basis for S, $\dim S = 1$

3. (6 points) Let $\mathcal{P}_2$ denote the vector space of polynomials of degree at most 2, and let $T\colon \mathcal{P}_2 \to \mathcal{P}_2$ be a linear transformation such that

$$T(1+x) = x, \quad T(x+x^2) = x^2 \quad \text{and} \quad T(x^2+1) = 1.$$

What is the value of $T(2+2x+2x^2)$?

(a) $1+x+x^2$ (b) 2 (c) $4x+x^2$ (d) $4+4x+4x^2$ (e) $x+x^2$

$2+2x+2x^2 = (1+x)+(x+x^2)+(x^2+1)$. (*)

so $T(2+2x+2x^2) = T(1+x) + T(x+x^2) + T(x^2+1)$

$= x + x^2 + 1$

4. (6 points) Let $\mathcal{P}_3$ denote the vector space of polynomials of degree at most 3. Let $T\colon \mathbb{R}^3 \to \mathcal{P}_3$ be a linear transformation. Which of the following statements is always correct?

I. T can not be one-to-one.
II. T can not be onto.
III. If T is one-to-one, then the dimension of range(T) is 3.
IV. rank(T) + nullity(T) = 4.

(a) II, III and IV. (b) II and IV. (c) I only. (d) III only. (e) II and III.

- $\dim \mathbb{R}^3 = 3$, $\dim P_3 = 4$.
- rank(T) + nullity(T) = $\dim \mathbb{R}^3 = 3$. So IV is false.
- T is one-to-one if nullity $(T)=0$, rank $(T)=3$; I possible. but not always true
- T is onto if rank$(T) = 4$, nullity $(T) = -1$, so II is true
- If nullity $(T)=0$, then rank $(T) = 3 = \dim(\text{range } (T))$, so III is always true.

5. (6 points) Let A, B and C be 2×2 matrices such that $\det(A) = 3$, $\det(B) = \sqrt{2}$ and C is invertible. What is the value of $\det(3C^{-1}AB^2C^T)$, where C^T denotes the transpose of C?

(a) 108 (b) 18 (c) 54 (d) 27 (e) 6

$$\det(3C^{-1}AB^2C^T) = 3^2 \cdot \det(C)^{-1} \cdot \det(A) \cdot \det(B)^2 \cdot \det(C^T)$$

$$= 9 \det(A) \det(B)^2 \quad \text{since } \det(C) = \det(C^T)$$

$$= 9 \cdot 3 \cdot \sqrt{2}^2 = 54.$$

6. (6 points) Let $\mathcal{P}_1$ denote the vector space of polynomials of degree at most 1. Consider the two bases $\mathcal{E} = \{1, x\}$ and $\mathcal{B} = \{1 - 2x, -1 + x\}$ of $\mathcal{P}_1$. Which of the following is the first row of the change of basis matrix $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$ from $\mathcal{E}$ to $\mathcal{B}$?

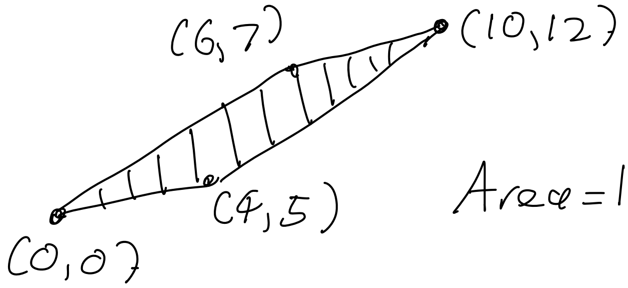
(a) $[-1 \quad -2]$ (b) $[-2 \quad -1]$ (c) $[1 \quad 1]$ (d) $[-1 \quad -1]$ (e) $[1 \quad -2]$

$$\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = \left[[1-2x]_{\mathcal{E}} \; [-1+x]_{\mathcal{E}} \right] = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}.$$

$$\underset{\mathcal{B} \leftarrow \mathcal{E}}{P} = \left(\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}\right)^{-1} = \frac{1}{-1} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

7. (6 points) Find the area of the parallelogram with vertices $(0,0)$, $(6,7)$, $(4,5)$ and $(10,12)$.

(a) 3 (b) 1 (c) 4 (d) 2 (e) 5



$$\text{Area} = \left| \det \begin{bmatrix} 6 & 7 \\ 4 & 5 \end{bmatrix} \right|$$
$$= |6 \cdot 5 - 4 \cdot 7|$$
$$= |2| = 2$$

8. (6 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 8 & 9 \end{bmatrix}.$$

What is the determinant of A?

(a) 0 (b) -4 (c) 4 (d) 3 (e) -3

$$\det \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 8 & 9 \end{vmatrix} = \det \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{vmatrix} = \det \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{vmatrix} = 1 \cdot 1 \cdot (-4) = -4$$

$R2 \to R2 - 2R1$

$R3 \to R3 - 3R3$

$R3 \to R3 - 2R3$

9. (15 points) Consider the bases of $\mathcal{P}_2$ given by

$$\mathcal{B} = \{4 + 3x + x^2, 1 + x - 2x^2, 1 - x^2\} \quad \text{and} \quad \mathcal{C} = \{1 + x, 1 + x^2, 1 - x^2\}.$$

$\mathcal{E} = \{1, x, x^2\}$ standard basis

(a) Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{C}$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 4 & 1 & 1 \\ 1 & 0 & 0 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & -1 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 4 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 1 & -2 & -1 \end{bmatrix}$$

$\underset{\mathcal{E}\leftarrow\mathcal{C}}{P}$ $\quad$ $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$

$$\downarrow \begin{bmatrix} 1 & 1 & 1 & 4 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & -2 & -1 \end{bmatrix}$$

$$\downarrow \begin{bmatrix} 1 & 1 & 1 & 4 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 & -2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 4 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \leftarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

I $\quad$ $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(b) Suppose that $p(x)$ is a polynomial in $\mathcal{P}_2$ with $[p(x)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. What is $[p(x)]_{\mathcal{C}}$?

$$[p(x)]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,[p(x)]_{\mathcal{B}} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}.$$

(c) What is $p(x)$ in (b)?

$$p(x) = 1\cdot(4+3x+x^2) + 1\cdot(1+x-2x^2) + 2(1-x^2)$$
$$= 7 + 4x - 3x^2$$

10. (14 points) Consider the matrix

$$A = \begin{bmatrix} s & -1 & 0 \\ -1 & s & -1 \\ 0 & -2 & s \end{bmatrix}$$

where s is a real number.

(a) Compute the determinant of A.

Cofactor expansion along row 1:

$$\det A = s \begin{vmatrix} s & -1 \\ -2 & s \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ 0 & s \end{vmatrix} + 0 \begin{vmatrix} -1 & s \\ 0 & -2 \end{vmatrix}$$

$$= s(s^2-2) - s$$

$$= s^3 - 3s$$

$$= s(s^2-3)$$

(b) For which values of s is the matrix A invertible?

A is invertible unless $s(s^2-3)=0$
ie for all s except $s=0$ and $s=\pm\sqrt{3}$.

(c) If the matrix A is invertible, give a formula for $\det(A^{-1})$ in terms of s.

$$\det(A^{-1}) = \frac{1}{\det A} = \frac{1}{s(s^2-3)}$$ if A^{-1} exists.

11. (15 points) Let $\mathcal{P}_3$ denote the vector space of polynomials of degree at most 3, and $T\colon \mathcal{P}_3 \to \mathbb{R}^2$ be the linear transformation given by

$$T(p(x)) = \begin{bmatrix} p(-1) \\ p(1) \end{bmatrix}$$

for $p(x)$ in $\mathcal{P}_2$ (you do not have to explain why T is a linear transformation).

(a) Write down the matrix $\underset{\mathcal{E}\leftarrow\mathcal{B}}{[T]}$ of T with respect to the standard bases

$$\mathcal{B} = \{1, x, x^2, x^3\} \text{ of } \mathcal{P}_3 \quad \text{and} \quad \mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \text{ of } \mathbb{R}^2.$$

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{[T]} = \Big[[T(1)]_{\mathcal{E}} \ [T(x)]_{\mathcal{E}} \ [T(x^2)]_{\mathcal{E}} \ [T(x^3)]_{\mathcal{E}} \Big]$$

$$= \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{E}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{\mathcal{E}} \begin{bmatrix} (-1)^2 \\ 1^2 \end{bmatrix}_{\mathcal{E}} \begin{bmatrix} (-1)^3 \\ 1^3 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = A$$

(b) Find a basis for the range of T.

$$A \to \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = B$$

Pivot columns of B are first and second.
First, second columns of A are a basis of col(A), $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. These are coordinate vectors with respect to $\mathcal{E}$ of a basis $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ of range T.

(c) Find a basis for the kernel of T. Make sure to write each element of your basis as a polynomial.

Null space of A is solution space of $\begin{cases} x_1 + x_3 = 0 \\ x_2 + x_4 = 0 \end{cases}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s \\ -t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ are a basis of null(A), and the coordinate vectors with respect to $\mathcal{B}$ of a basis $\{x^2-1, x^3-x\}$ of ker(T).

12. (14 points) Consider the system of equations

$$\begin{cases} sx_1 & - & 3x_2 & = 2 \\ 3x_1 & - & sx_2 & = 3 \end{cases}$$

where s is a number.

(a) For values of s for which the system has a unique solution, USE CRAMER'S RULE to find the solution, giving explicit formulas for x_1 and x_2 in terms of s.

$$\underbrace{\begin{bmatrix} s & -3 \\ 3 & -s \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{\vec{b}}$$

$\det A = -s^2+9$. When $\det A \neq 0$, the system has a unique solution

$$x_1 = \frac{\begin{vmatrix} 2 & -3 \\ 3 & -s \end{vmatrix}}{-s^2+9} = \frac{-2s+9}{-s^2+9} = \frac{2s-9}{s^2-9}$$

$$x_2 = \frac{\begin{vmatrix} s & 2 \\ 3 & 3 \end{vmatrix}}{-s^2+9} = \frac{3s-6}{-s^2+9} = -\frac{3s-6}{s^2-9}.$$

(b) For which values of s does the system have a unique solution?

There is a unique solution when $-s^2+9 \neq 0$ ie when $s \neq \pm 3$.