Name: **VERSION: 1**

Section #: Carl Frederick Gauss

Exam 3, Math 20580
April 17,2025

- The Honor Code *is* in affect for this examination. All work is to be your own.
- Please turn off and put away all cellphones and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for one hour and 15 minutes.
- Be sure that your name and your instructor's name and your section number are on the front page of your exam.

01	Matthew Dyer	8:20-9:10	127 HH
02	Matthew Dyer	9:25-10:15	127 HH
03	Jeffrey Diller	10:30-11:20	136 DBRT
04	Han Lu	11:30-12:20	114 PASQ
05	Sudipta Gosh	12:50-1:40	140 DBRT
06	Sudipta Gosh	2:00-2:50	140 DBRT
07	Claudia Polini	3:30-4:20	140 DBRT

- There are 12 problems: 8 are multiple choice and 4 are partial credit.
- Be sure that you have all 9 pages of this exam.
- Each multiple choice question is worth 6 points. Your score will be the sum of the best 7 scores on the multiple choice plus your score on questions 9–12.
- Make sure you have correctly marked your multiple choice answers by the time the exam ends. No credit will be given for incorrectly marked answers.

Please mark you answers with an **X**, not a circle.

	1.	(a)	(b)	(•)	(d)	(e)
	2.	(a)	(b)	(c)	(•)	(e)
				1		
	3.	(a)	(b)	(c)	(d)	(•)
− >	4	(a)	(•)	(c)	(d)	(e)
				2		
− >	5.	(•)	(b)	(c)	(d)	(e)
	6.	(a)	(b)	(c)	(d)	(•)
				3		
	7.	(a)	(b)	(•)	(d)	(e)
	8.	(a)	(b)	(c)	(•)	(e)
				4		

MC Total.	
9.	
10.	
11.	
12.	
Total.	

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			3		
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8.	(a)	(b)	(c)	(d)	(e)
			4		

MC Total.	________
9.	________
10.	________
11.	________
12.	________
Total.	________

Multiple choice problems

1. (6 points) The eigenvalues of a certain 3×3 matrix A are -1, 0, and 1. Determine which statement is true.

(a) The linear transformation associated to A is onto.

(b) $\operatorname{rank}(A) = 1$

(c) $\det(A) = 0$

(d) A is not diagonalizable.

(e) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Since 0 is an eigenvalue, there is $0 \neq \vec{v}$ in $\mathbb{R}^3$ so $A\vec{v} = 0\vec{v} = \vec{0}$. Hence $\vec{v}$ is in null(A), so A is not invertible, and det(A)=0.

2. (6 points) Suppose that we find a QR decomposition of $A = \begin{bmatrix} 1 & -1 & 2 & -5 \\ 1 & 3 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ -1 & -3 & 1 & 3 \end{bmatrix}$ with

$$Q = \begin{bmatrix} 1/2 & -1/2 & \sqrt{2}/2 & 0 \\ 1/2 & 1/2 & 0 & \sqrt{2}/2 \\ -1/2 & 1/2 & \sqrt{2}/2 & 0 \\ -1/2 & -1/2 & 0 & \sqrt{2}/2 \end{bmatrix}.$$

What is the first entry in the third row of R?

(a) -2 (b) -1 (c) 2 (d) 0 (e) 1

Since R is upper triangular, all entries below the main diagonal are zero. Hence the first entry in the third row is zero.

Alternatively, just compute the appropriate entry of $R = Q^T A$, to find it is zero.

3. (6 points) Suppose that A is a 3×3 matrix, that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for A with eigenvalue -1, and that $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for A with eigenvalue 2. What is $A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$?

(a) $\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} -3 \\ -1 \\ -3 \end{bmatrix}$

$A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. So

$A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = A\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - A\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ -3 \end{bmatrix}$

4. (6 points) Let S be a one dimensional subspace (i.e. a line through the origin) in $\mathbb{R}^2$. Let A be the 2×2 matrix such that $A\vec{x}$ is the orthogonal projection of $\vec{x}$ onto S. Which of the following are true statements?

(I) Every vector $\vec{v}$ in S is an eigenvector of A.
(II) Every vector in the orthogonal complement of S is an eigenvector of A.
(III) 0 and 1 are eigenvalues of A.

(a) (I) and (III) only (b) (I), (II) and (III) (c) (I) and (II) only (d) none of them
(e) (III) only

$A\vec{x} = \text{Proj}_S(\vec{x})$ for $\vec{x}$ in $\mathbb{R}^2$.

If $\vec{x}$ is in S, $\text{Proj}_S(\vec{x}) = \vec{x}$ so $A\vec{x} = \vec{x} = 1\cdot\vec{x}$ and $\vec{x}$ is an eigenvector of A with eigenvalue 1.

If $\vec{x}$ is in $S^\perp$, then $\vec{x} = \text{Perp}_S(\vec{x}) = \vec{x} - \text{Proj}_S(\vec{x})$ so $\text{Proj}_S(\vec{x}) = \vec{x} - \vec{x} = \vec{0}$. Then $A\vec{x} = 0\vec{x}$, so $\vec{x}$ is an eigenvector of A with eigenvalue 0.

Since $\dim(S) + \dim(S^\perp) = 2$ and $\dim S = 1$, we have $\dim(S^\perp) = 1$. A non-zero vector in S (respectively) $S^\perp$ is an eigenvector of A with eigenvalue 1 (resp., 0). So both 0 and 1 are eigenvalues of A.

5. (6 points) Find the projection of the vector $\vec{v} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ on the subspace W of $\mathbb{R}^3$ spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

(a) $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

Note $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0$ so $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ are an orthogonal basis of their span, W.

So $\operatorname{Proj}_W\left(\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}\right) = \dfrac{\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \dfrac{\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$= \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{-2}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$

6. (6 points) What is the least squares solution $\vec{x}$ of the linear system $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$?

(a) no solution (b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Write the system as $A\vec{x} = \vec{b}$. The normal equations are $A^T A \vec{x} = A^T \vec{b}$.

$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix}$

$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. So

$\vec{x} = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \frac{1}{6 \cdot 3 - 3 \cdot 3} \begin{bmatrix} 3 & -3 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -9 \\ 18 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

7. (6 points) Which of the following statements about the differential equation

$$y' = y^2$$

are true?

(I) The equation is linear.
(II) The equation is separable.
(III) The equation is autonomous.

(a) (III) only (b) (I) and (III) only (c) (II) and (III) only (d) (I) and (II) only

(e) None of them

$\frac{dy}{dx} = y^2$. Since y^2 does not depend on x, the equation is autonomous. It can be written $\frac{1}{y^2}dy = 1 \cdot dx$ so is separable. It cannot be written

$\frac{dy}{dx} + P(x)y = g(x)$, so is not linear.

8. (6 points) Let $y(x)$ be the solution of the initial value problem

$$\frac{dy}{dx} = 2xy^2, \quad y(0) = 2.$$

What is $y(1)$?

(a) 1 (b) 1/3 (c) 2 (d) -2 (e) 3/2

Separate: $\frac{1}{y^2}dy = 2x\,dx$

Integrate: $-\frac{1}{y} = x^2 + C.$

When $x=0$, $y=2$: $-\frac{1}{2} = C.$

So $-\frac{1}{y} = x^2 - \frac{1}{2} = \frac{2x^2-1}{2}$, $y = \frac{2}{1-2x^2}$

When $x=1$, $y = \frac{2}{-1} = -2.$

Partial credit problems

9. (18 points) Consider the matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(a) Find the eigenvalues of A

$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix}$

expand down column 1

$= (3-\lambda)[(2-\lambda)\cdot -\lambda - 1\cdot(-1)] = (3-\lambda)(\lambda^2-2\lambda+1) = (3-\lambda)(\lambda-1)^2$

Eigenvalues 1, 1, 3.

(b) Find a basis for each eigenspace of A.

$A-3I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{cases} x_1 = r \\ x_2 = 0 \\ x_3 = 0 \end{cases}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ basis for E_3

$A-1I = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{cases} x_1 = -\frac{1}{2}r \\ x_2 = r \\ x_3 = r \end{cases}, \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$ basis for E_1

(c) What are the *algebraic* multiplicities of the eigenvalues of A?

1 has alg. mult. 2

3 has alg. mult 1.

(d) What are the *geometric* multiplicities of the eigenvalues of A?

1 has geom. mult 1

3 has geom mult 1.

(e) Diagonalize A if possible. That is, find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. If A is not diagonalizable, explain why not.

A is not diagonalizable since the geom. mult of the eigenvalue 1 is not equal to the alg. mult. of 1.

10. (12 points) Consider the matrix

$$A = \begin{bmatrix} 3 & -4 & 4 \\ 0 & 0 & 2 \\ 1 & 2 & -2 \\ 0 & 1 & 10 \end{bmatrix}$$

Use the Gram-Schmidt process to find an *orthonormal* basis for Col(A).

Write $A = [\vec{x_1}\ \vec{x_2}\ \vec{x_3}]$ where $\vec{x_i}$ is the i-th column of A.

Take $v_1 = \vec{x_1}$

$$\vec{v_2} = \vec{x_2} - \frac{\vec{x_2} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} = \begin{bmatrix} -4 \\ 0 \\ 2 \\ 1 \end{bmatrix} - \frac{-10}{10} \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

$$\vec{v_3} = \vec{x_3} - \frac{\vec{x_3} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} - \frac{\vec{x_3} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} \vec{v_2} = \begin{bmatrix} 4 \\ 2 \\ -2 \\ 10 \end{bmatrix} - \frac{10}{10} \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{0}{11} \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 10 \end{bmatrix}$$

$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is an orthogonal basis for col(A).

Dividing each vector $\vec{v_i}$ by its length gives an orthonormal basis

$$\frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{114}} \begin{bmatrix} 1 \\ 2 \\ -3 \\ 10 \end{bmatrix}$$

of col(A).

11. (14 points)

(a) Find the solution $y(x)$ of the initial value problem

$$x\frac{dy}{dx} + 3y = 2x, \quad y(1) = 2$$

$(*)$: $\dfrac{dy}{dx} + \dfrac{3}{x}y = 2$. We consider this for $x > 0$

An integrating factor is

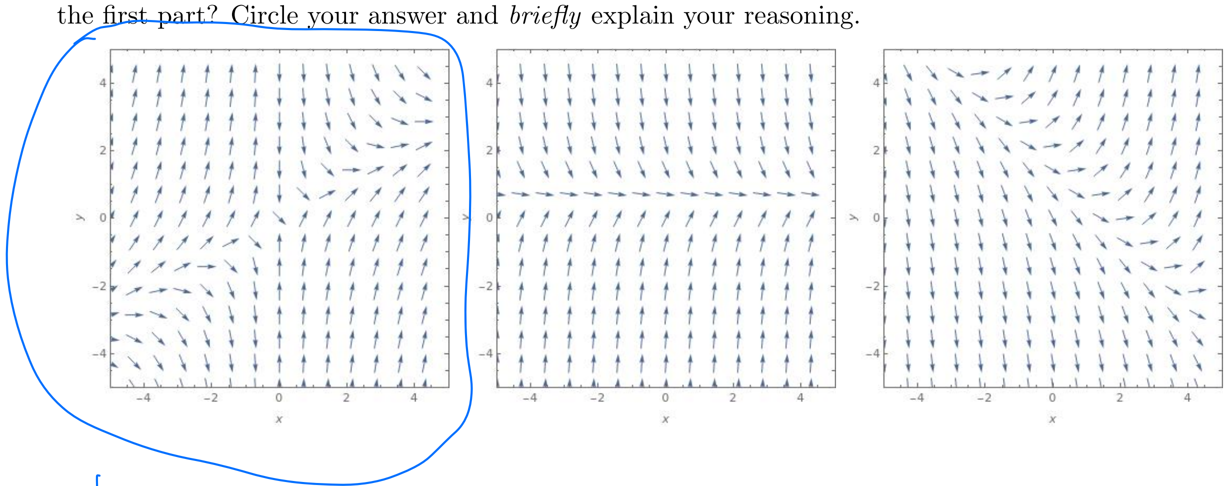
$\mu(x) = e^{\int \frac{3}{x} dx} = e^{3\ln|x|} = |x|^3 = x^3$.

$x^3(*)$: $\dfrac{d}{dx}(x^3 y) = 2x^3$. Integrating, $x^3 y = \frac{1}{2}x^4 + C$.

When $x=1$, $y=2$, so $1^3 \cdot 2 = \frac{1}{2}\cdot 1^4 + C$, $C = \frac{3}{2}$,

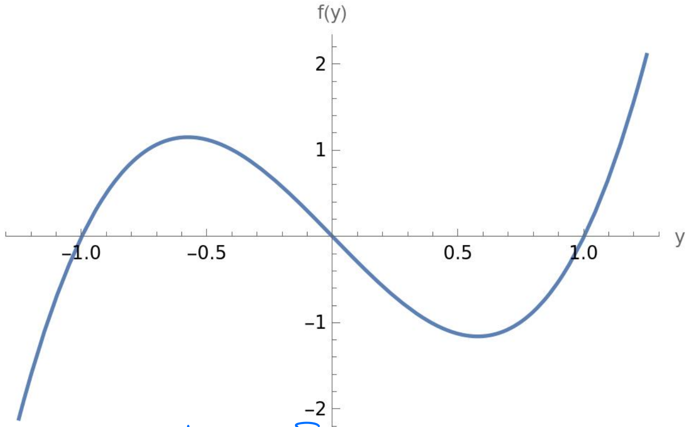
$y = \dfrac{1}{x^3}\left[\frac{1}{2}x^4 + \frac{3}{2}\right] = \frac{1}{2}x + \dfrac{3}{2x^3}$

(b) Which of the following three pictures shows the slope (i.e. direction) field for the differential equation in the first part? Circle your answer and *briefly* explain your reasoning.



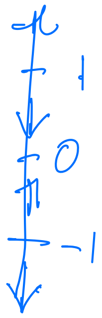
$\dfrac{dy}{dx} = 2 - 3\dfrac{y}{x}$. On the line $y = mx$, the direction vectors should all have slope $2-3m$ eg slope -1 on $y=x$ and slope 5 on $y=-x$. Only the first graph has direction vectors with positive slopes on $y=x$ for instance. Another way is to note that if $\frac{y}{x} < 0$, the direction vectors should all have positive slope

12. (14 points) Consider the autonomous ODE $\frac{dy}{dx} = f(y)$ where the graph of $f(y) = 3y^3 - 3y$ is shown below.



(a) Find and classify the critical points of ~~f~~ the ODE justifying your classification *briefly* in words or by drawing a phase portrait (i.e. phase line).

The critical points of the ODE are the-values c such that $f(c)=0$ ie $c=-1, 0, 1$ from the graph.
The phase line is

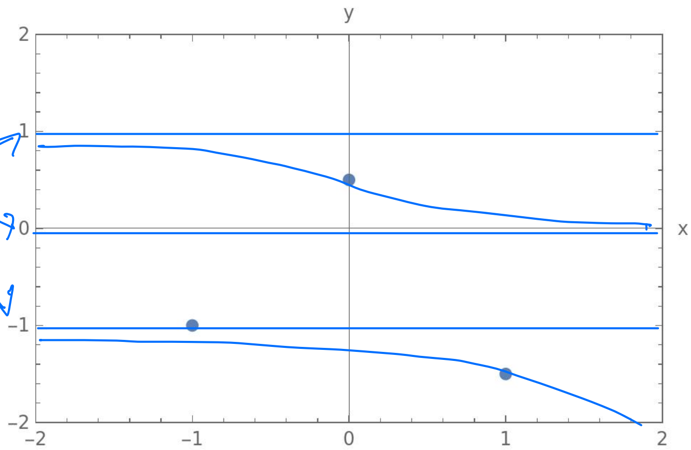


so $y=1$ is unstable equilibrium
$y=0$ is stable equilibrium
$y=-1$ is unstable equilibrium.

(Points were also given for finding/classifying the critical points of f, as you learned in Calc I.).

(b) On the graph below, give rough sketches of the (three) solutions $y(x)$ of the differential equation that pass through the indicated points $(x, y) = (-1, -1)$, $(0, \frac{1}{2})$, $(1, -\frac{3}{2})$.

equilibrium solutions



The three solutions are those drawn through the blue dots.