Math 20580 Name: Solutions

Practice Midterm 1 Instructor:

February 12, 2015 Section:

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an × through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. ~~a~~ b c d e
2. a b c ~~d~~ e
3. a b c d ~~e~~
4. a b ~~c~~ d e
5. a b ~~c~~ d e
6. ~~a~~ b c d e
7. a b ~~c~~ d e
8. a b c d ~~e~~

Part I: Multiple choice questions (7 points each)

1. Consider the linear system

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned}$$

Which of the following (x_1, x_2, x_3) is a solution?

(a) $(-3/5, 7/5, 1)$ (b) $(2/5, 3/5, 1)$ (c) $(7/5, 3/5, 1)$ (d) $(4/5, -3/5, 1)$
(e) $(2/5, 7/5, 1)$

Augmented matrix is

$$\begin{pmatrix} 1 & 4 & 0 & 5 \\ 2 & 3 & -2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 4 & 0 & 5 \\ 0 & -5 & -2 & -9 \end{pmatrix}$$

So $\left.\begin{aligned} -5x_2 &= 2x_3 - 9 \\ x_1 + 4x_2 &= 5 \end{aligned}\right\}$ x_3 free. Setting $x_3 = 1$ gives $x_2 = 7/5$, $x_1 = 5 - 28/5 = -3/5$.

2. For which constants t do the vectors $\begin{bmatrix} 1 \\ 0 \\ t \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$ span all of $\mathbb{R}^3$?

(a) $t = 1$ only (b) all $t \neq 1$ (c) $t = -1/5$ only (d) all $t \neq -1/5$
(e) there are no t

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 3 & -2 & t \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & -2 \\ 0 & -8 & t-3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2/5 \\ 0 & -8 & t-3 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2/5 \\ 0 & 0 & t-3+\frac{16}{5} \end{pmatrix}$$

So span if matrix eqn. always consistant,

i.e. $t - 3 + \frac{16}{5} = t + 1/5 \neq 0$

i.e. $t \neq -1/5$.

3. Which column in the matrix below is the first from the left which is a linear combination of the previous ones?

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 1 & 0 & 2 & 0 & 3 \end{bmatrix}$$

(a) $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

?

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 1 & 0 & 2 & 0 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & +\text{[illegible]} & +1 & +2 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

No pivot only a last column.

4. Is the linear transformation corresponding to the matrix below one-to-one or onto?

$$\begin{bmatrix} 2 & 1 & 3 & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

(a) both one-to-one and onto (b) one-to-one but not onto
(c) onto but not one-to-one (d) neither one-to-one nor onto

$$\rightsquigarrow \begin{pmatrix} 2 & 1 & 3 & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 3 & 2 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & 1 & 2 \\ 0 & 1 & -1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -3 & 1 & 2 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 5 \end{pmatrix}$$

No rows all 0 $\Rightarrow$ onto.

No pivot in col. 4 $\Rightarrow$ not 1-to-1

5. Find the determinant of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 8 & 2 & 9 \\ 0 & 3 & 0 & 4 \\ -1 & 10 & 20 & 30 \end{bmatrix}$$

(a) 2 (b) 4 (c) -4 (d) -2 (e) 0

$$\det A = -\det\begin{pmatrix} -1 & 10 & 20 & 30 \\ 0 & 8 & 2 & 9 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{pmatrix} = \det\begin{pmatrix} -1 & 10 & 20 & 30 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 8 & 2 & 9 \end{pmatrix}$$

$$= \det\begin{pmatrix} -1 & 10 & 20 & 30 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & -7 \end{pmatrix} = -\det\begin{pmatrix} -1 & 10 & 20 & 30 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & -7 \\ 0 & 0 & 0 & -2 \end{pmatrix} = -4$$

6. Which of the following matrices are invertible?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 5 & 2 \\ 3 & 6 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 10 & 20 \\ 0 & 0 & 9 \end{bmatrix}$$

(a) only D (b) they are all invertible (c) C and D only
(d) B and C only (e) only C

A not square

B has row all 0 so not onto

C has linearly dependent columns (first and last same)

D has det = 90 so invertible

7. If B is the matrix below, and $C = (B^T)^5$, compute $\det(C)$.

$$B = \begin{bmatrix} 0 & 6 & 2 \\ 2 & 8 & 7 \\ 0 & 2 & 1 \end{bmatrix}$$

(a) 0 (b) 2^{10} (c) -2^{10} (d) 6^5 (e) -6^5

$$\det C = \det(B^T)^5 = (\det B)^5$$

$$B \rightsquigarrow \begin{pmatrix} 2 & 8 & 7 \\ 0 & 2 & 1 \\ 0 & 6 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 8 & 7 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{So } \det B = -4$$

$$(-4)^5 = -4^5 = -2^{10}.$$

8. Compute the dimension of the Null-space of the matrices below.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$(\dim \mathrm{Nul}(A), \dim \mathrm{Nul}(B), \dim \mathrm{Nul}(C)) =$

(a) $(0,0,2)$ (b) $(2,2,2)$ (c) $(1,0,3)$ (d) $(0,1,1)$ (e) $(2,1,1)$

dim of Nul = # columns without pivots

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$

So $\dim \mathrm{Nul}(C) = 1$

Part II: Partial credit questions (11 points each). Show your work.

9. Find a solution to the linear system

$$\begin{aligned} x_1 + x_3 &= 1 \\ 2x_1 + 2x_3 + x_4 &= 1 \\ x_1 + x_2 + 2x_3 &= 2 \\ 2x_2 + x_3 + x_4 &= 1 \end{aligned}$$

Augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 & 2 \\ 0 & 2 & 1 & 1 & 1 \end{array}\right) \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

So $x_4 = -1$

$-x_3 + x_4 = -1 \implies x_3 = 0$

$x_2 + x_3 = 1 \implies x_2 = 1$

$x_1 + x_3 = 1 \implies x_1 = 1$

10.

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 1 & 1 \\ 3 & 1 & 0 & -1 \end{bmatrix}$$

(a) A gives a linear transformation $T_A : \mathbb{R}^p \to \mathbb{R}^q$. What are the numbers p and q?

$p = 4,\ q = 3$

(b) Find a nonzero vector x in $\mathbb{R}^p$ which is a solution of the homogeneous equation $Ax = 0$ (or explain why there are none).

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 1 & 1 \\ 3 & 1 & 0 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & -3 & -1 \\ 0 & -2 & -6 & -4 \end{pmatrix}$$

So x_4 free, can set $x_4 = 1$.

$$\rightsquigarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & -3 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$

this gives $x_3 = -1,\ x_2 = 1,\ x_1 = 0$

ie. soln. $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$

(c) Find a vector b in $\mathbb{R}^q$ which does not lie in the image of T_A (or explain why there are none).

The echelon form has no rows all zero.

Therefore T_A is onto and there are no bs.

11. Find the inverse of A, where

$$A = \begin{bmatrix} 2 & -1 & 4 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 4 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & -1 & 4 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 4 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 4 & 10 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{array}\right)$$

& inverse is $\begin{pmatrix} 0 & 0 & 1 \\ -1 & 4 & 10 \\ 0 & 1 & 2 \end{pmatrix}$

12. Consider the matrix A below.

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 4 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

(a) Find a basis for the column space Col(A).

$$\rightsquigarrow \begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix}$$

See pivots in 1st three columns so a basis is

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}.$$

(b) Let v be the last column of A. Find the coordinates of v relative to the basis found in part (a).

Solving the linear system $x_3 = 1/3$, $x_2 = 0$, $x_1 = 1/3$.

So $$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{pmatrix} 1/3 \\ 0 \\ 1/3 \end{pmatrix}.$$