1. Which of the following statements is always true?

I. The null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.

II. If the set $B = \{v_1, \ldots, v_n\}$ spans a vector space V and $\dim V = n$, then B is a basis for V.

III. The rank of a $m \times n$ matrix A is the dimension of the column space of A.

(a) II and III (b) I (c) I and III (d) I and II and III (e) None are true.

Solution. II and III are true.
I. The null space is a subspace of $\mathbb{R}^n$ so this is false.
II. This is the basis theorem.
III. This is the definition of rank.

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2. Let $\mathbb{P}_n$ be the space of polynomial functions of degree at most n. Which of the following is NOT true?

(a) The set of all polynomials of degree less than 2 with $f(0) = 1$ is a subspace of $\mathbb{P}_3$.
(b) $\mathbb{P}_5$ is a subspace of $\mathbb{P}_8$.
(c) If $T : \mathbb{P}_4 \to \mathbb{R}$ is a linear transformation, then the kernel of T is a subspace of $\mathbb{P}_4$.
(d) If $\mathcal{B}$ is a basis for $\mathbb{P}_2$, then $\mathcal{B}$ is linearly independent in $\mathbb{P}_3$
(e) $\mathcal{B} = \{t - 1, t + 1, t^2\}$ is a basis for $\mathbb{P}_2$.

Solution. The set of all polynomials of degree less than 2 with $f(0) = 1$ is not closed under addition, so it cannot be a subspace.

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3. Which of the following is NOT a linear transformation?

(a) $T : \mathbb{R} \to \mathbb{R}$ where $T(x) = x^2$
(b) $D : \mathbb{P}_5 \to \mathbb{P}_4$ defined by $D(f(t)) = 2f'(t)$, where f' denotes the derivative of f.
(c) $T : C \to C$ where $T(f(t)) = (t + 1)f(t)$ and C is the space of continuous real valued functions.
(d) $T : V \to V$ that sends each vector to itself (i.e. $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$).
(e) $T : \mathbb{P}_3 \to \mathbb{R}$ where $T(f(t)) = 5f(1)$

Solution. If $T(x) = x^2$, then $T(5x) = 25x^2 \neq 5x^2 = 5T(x)$, so T is not a linear transformation

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4. The transformation $T : \mathbb{P}_3 \to \mathbb{R}$ defined by $T(f(t)) = f(2)$ is a linear transformation. What is a basis for the kernel of T?

(a) $\{t^3 - 8, t^2 - 4, t - 2\}$ (b) $\{1, 2, 4, 8\}$ (c) $\{1, t, t^2, t^3\}$ (d) $\{\begin{bmatrix}1\end{bmatrix}\}$ (e) $\{t, t^2, t^3\}$

Solution. Using the basis $\{1, t, t^2, t^3\}$ for $\mathbb{P}_3$, we see that the matrix for the transformation is $\begin{bmatrix}1 & 2 & 4 & 8\end{bmatrix}$. The matrix is in row reduced echelon form, so variables 2,3,4 are free. Therefore, a basis is $\left\{ \begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-4\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-8\\0\\0\\1\end{bmatrix} \right\}$ or $\{t^3 - 8, t^2 - 4, t - 2\}$.

5. $\mathcal{B} = \{1+t, 1-t, 2+t+3t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $p(t) = t^2$ with respect to $\mathcal{B}$.

(a) $\begin{bmatrix} -1/2 \\ -1/6 \\ 1/3 \end{bmatrix}$ (b) $\begin{bmatrix} 1/2 \\ 1/6 \\ -1/3 \end{bmatrix}$ (c) $[1]$ (d) $\begin{bmatrix} 1+t \\ 1-t \\ 2+t+3t^2 \end{bmatrix}$ (e) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Solution. Row reduce $\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & -1/6 \\ 0 & 0 & 1 & 1/3 \end{bmatrix}$ thus the coordinate vector is $\begin{bmatrix} -1/2 \\ -1/6 \\ 1/3 \end{bmatrix}$.

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6. Find a basis for the eigenspace of $\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$ with eigenvalue 3.

(a) $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$ (c) $\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$ (d) $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$
(e) 3 is not an eigenvalue of the matrix

Solution. If $A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$, then the solutions of $(A - 3I)\mathbf{x} = \mathbf{0}$ are all multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

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7. Let A be the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -3 & * & -1 \end{bmatrix},$$

where $*$ can be any number. What are the eigenvalues of A?

(a) $2, 3, -1$ (b) $2, 1, -3$ (c) It is not possible to tell without knowing $*$ (d) $2, 0, 0$
(e) $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$

Solution. The matrix is lower triangular, so the characteristic polynomial is $(2-t)(3-t)(-1-t)$, and the eigenvalues are $2, 3, -1$.

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8. What is the characteristic polynomial of the matrix $\begin{bmatrix} -3 & 4 \\ 1 & 2 \end{bmatrix}$?

(a) $\lambda^2 + \lambda - 10$ (b) $\lambda^2 + \lambda - 6$ (c) $\lambda^2 - \lambda + 10$ (d) $(-3-\lambda)(2-\lambda)$ (e) $\lambda^2 - \lambda + 6$

Solution. The characteristic polynomial is

$$\det \begin{bmatrix} -3-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} = (-3-\lambda)(2-\lambda) - 4 = \lambda^2 + \lambda - 10.$$

9. The eigenvalues of a 3×3 matrix A are $-1, 1$ with 1 having multiplicity 2. Which of the following statements is true?

(a) The equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$.
(b) The matrix A has a 3-dimensional null space.
(c) The diagonal entries of A are $-1, 1, 1$. (d) The matrix A can be diagonalized.
(e) The equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

Solution. If none of the eigenvalues are zero then the matrix is invertible.

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10. Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ be two bases of $\mathbf{R}^2$.

(a) Find the change of coordinates matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$.

(b) Suppose that $\mathbf{v}$ is a vector in $\mathbf{R}^2$ such that $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Compute $[\mathbf{v}]_{\mathcal{C}}$.

Solution.

(a) We row reduce:

$$\begin{bmatrix} -1 & -2 & 2 & -3 \\ 2 & 3 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & -1 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 8 & -1 \\ 0 & 1 & -5 & 2 \end{bmatrix},$$

so the change of basis matrix is

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 8 & -1 \\ -5 & 2 \end{bmatrix}.$$

(b)

$$[\mathbf{v}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P}[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 8 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 7 \end{bmatrix}.$$

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11. Consider the transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ given by $T(f(t)) = (3t - 2)f'(t)$, where $f'(t)$ is the derivative of $f(t)$. For example,

$$T(t^2 - 3t + 1) = (3t - 2)(2t - 3) = 6t^2 - 13t + 6.$$

The function T is a linear transformation. You do not need to show this.

(a) Calculate the matrix A of T with respect to the basis $\mathcal{B} = \{1, t, t^2\}$ for $\mathbb{P}_2$.
(b) Find a basis for the kernel (null space) of T consisting of elements of $\mathbb{P}_2$. What is its dimension?
(c) Find a basis for the range of T consisting of elements of $\mathbb{P}_2$. What is its dimension?

Solution.

(a) We compute

$$\begin{aligned} T(1) &= 0, \\ T(t) &= 3t - 2, \\ T(t^2) &= (3t-2)(2t) = 6t^2 - 4t. \end{aligned}$$

It follows that the matrix of T with respect to the basis $\mathcal{B}$ is

$$A = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 3 & -4 \\ 0 & 0 & 6 \end{bmatrix}.$$

(b) Row reducing A we have

$$\begin{bmatrix} 0 & -2 & 0 \\ 0 & 3 & -4 \\ 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis for the null space of A is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, which corresponds to $\{1\}$ in $\mathbb{P}_2$. The dimension is 1.

(c) The range of T is the same as the column space of A, which has basis $\left\{ \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 6 \end{bmatrix} \right\}$, which corresponds to $\{-2 + 3t, -4t + 6t^2\}$ in $\mathbb{P}_2$. The dimension is 2.

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12. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix}$. Find a diagonal matrix D and a matrix P such that $A = PDP^{-1}$. You do not need to compute P^{-1}.

Solution. First compute the characteristic polynomial:

$$\det\begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 6 & 2-\lambda \end{bmatrix} = (2-\lambda)(3-\lambda)(2-\lambda)-(2-\lambda)6 = (2-\lambda)(\lambda^2-5\lambda) = \lambda(2-\lambda)(\lambda-5).$$

The eigenvalues are $0, 2, 5$.

We next find corresponding eigenvectors.

<u>eigenvalue 0:</u>

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis of the null space is $\left\{ \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\}$.

<u>eigenvalue 2:</u>

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis for the null space is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

<u>eigenvalue 5:</u>

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the null space is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

We can choose D, P so that

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -3 & 0 & 2 \end{bmatrix}.$$
