Name: Date: 04/03/2025

Math 20580 (L.A. and D.E.) Tutorial
Quiz Week 8

1. Apply the Gram-Schmidt process to the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

to obtain an orthonormal basis (a basis that is an orthogonal set of unit vectors) of $\mathbb{R}^3$.

Solution: Let

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix},$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \operatorname{proj}_{\mathbf{v}_1} \mathbf{x}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix},$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \operatorname{proj}_{\mathbf{v}_1} \mathbf{x}_3 - \operatorname{proj}_{\mathbf{v}_2} \mathbf{x}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} \frac{46}{9} \\ \frac{46}{9} \\ \frac{23}{9} \end{bmatrix}.$$

Then the wanted orthonormal basis is

$$\mathbf{e}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix},$$

$$\mathbf{e}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix},$$

$$\mathbf{e}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$