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Math 20580 (L.A. and D.E.) Tutorial
Worksheet 1

1. Consider the following eight 2×4-matrices.

$$A = \begin{bmatrix} 1 & -6 & 2 & -3 \\ -3 & 4 & -2 & -7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 2 & -3 \\ 0 & 0 & -4 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -3 & 2 & -3 \\ -3 & 4 & -2 & -7 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 2 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -3 & -2 & -5 & -8 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$G = \begin{bmatrix} -3 & -4 & -2 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Circle below the names X of those matrices which are in **row echelon form** (REF) and double circle the names Y of those matrices which are in **reduced row echelon form** (RREF), as in X and Y:

A B C D E F G H

(b) For those matrices $A, \ldots, H$ which are in row echelon form, circle their pivot entries and state their rank.

matrix	rank
B	2
D	1
F	2
G	1
H	0

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2. Use Gaussian elimination to solve the following system of linear equations. What are the ranks of the coefficient matrix and augmented matrix of the system?

$$\begin{cases} 2x + y - z = 1 \\ x + 2y - z = 2 \\ 3x + 2y + z = 10 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & 2 & -1 & 2 \\ 3 & 2 & 1 & 10 \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & -1 & 1 \\ 3 & 2 & 1 & 10 \end{array}\right]$$

$$\xrightarrow[R3 \to R3 - 3R1]{R2 \to R2 - 2R1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 1 & -3 \\ 0 & -4 & 4 & 4 \end{array}\right] \xrightarrow{R3 \leftrightarrow R4} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -4 & 4 & 4 \\ 0 & -3 & 1 & -3 \end{array}\right] \xrightarrow{R2 \to -\frac{1}{4}R2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -3 & +1 & -3 \end{array}\right]$$

$$\xrightarrow{R3 \to R3 + 3R2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -2 & -6 \end{array}\right] \leftarrow \text{REF.}$$

The linear system corresponding to the REF is

$$\begin{cases} x + 2y - z = 2 \\ \quad\; y - z = -1 \\ \quad\;\; -2z = -6. \end{cases}$$

Solving by back substitution gives

$$z = \frac{-6}{-2} = 3,$$

$$y = -1 + z = 2,$$

$$x = 2 - 2y + z = 1$$

The unique solution is

$$\begin{cases} x = 1 \\ y = 2 \\ z = 3 \end{cases}$$

The coefficient matrix has rank 3 (3 pivots in first 3 columns). The augmented matrix has rank 3 (since 3 pivots).

Note that in the row reduction, we did some of the EROs to avoid introducing fractions earlier than necessary.

The pivot positions of the REF are circled in red.

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3. Use Gauss-Jordan elimination to solve the following system of linear equations. What are the ranks of the coefficient matrix and augmented matrix of the system?

$$\begin{cases} x+y-z=0 \\ 3x+y-2z=0 \\ x-3y+z=0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 3 & 1 & -2 & 0 \\ 1 & -3 & 1 & 0 \end{array}\right] \xrightarrow[R3 \to R3-R1]{R2 \to R2-3R1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R3 \to R3-2R2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R2 \to -\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R1 \to R1-R2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow \text{RREF.}$$

The corresponding linear system is

$$\begin{cases} x \quad -\frac{1}{2}z=0 \\ y -\frac{1}{2}z=0 \\ 0=0. \end{cases}$$

The solution is $\begin{cases} x=\frac{1}{2}z \\ y=\frac{1}{2}z \\ z \text{ arbitrary} \end{cases}$. Writing $z=r$ for some real number r, $\begin{cases} x=\frac{1}{2}r \\ y=\frac{1}{2}r \\ z=r \end{cases}$ or in parametric vector form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \\ \frac{1}{2}r \\ \frac{1}{2}r \end{bmatrix} \text{ i.e. } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Rank of coefficient matrix: 2 (2 pivots in first 3 columns)
augmented matrix: 2 (2 pivots).

The pivot positions of the RREF are circled.

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4. Given the augmented matrices below corresponding to some systems of linear equations, determine how many solutions the systems have **(if any)**. Also, if the corresponding linear system is consistent, determine the number of free variables it has.

(a) $\left[\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 6 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & -3 & -6 & 0 & 0 \\ 0 & 2 & -7 & 5 & 2 \end{array}\right]$ (b) $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & -7 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 4 & 6 \end{array}\right]$

(c) $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 5 \\ 0 & 1 & -2 & 1 & -4 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 2 & 3 & 6 \end{array}\right]$

(a) $\xrightarrow{R_3 \to R_3 + 3R_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 6 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 2 & -7 & 5 & 2 \end{array}\right]$ The third row corresponds to the equation

$0x_1 + 0x_2 + 0x_3 + 0x_4 = 3$ ie $0 = 3$ which is false. The system has no solutions i.e it is inconsistent.

(b) $\xrightarrow{R_4 \to R_4 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & -7 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ REF

x_1, x_2, x_3 are bound ("pivot") variables, the other variable x_4 is a "free" variable. Solving by back-substitution, we see that each of of the free variables can be expressed in terms of the bound variable x_4,

$$\begin{cases} x_1 \quad -2x_3 + 3x_4 = -7 \\ x_2 \quad -2x_3 = 3 \\ x_3 + 2x_4 = 3 \end{cases} \qquad \begin{cases} x_1 = 2x_3 - 3x_4 - 7 = -1 - 7x_4 \\ x_2 = 3 + 2x_3 = 9 - 4x_4 \\ x_3 = 3 - 2x_4 \end{cases}$$

The (unique) free variable x_4 can take any value; there are infinitely many solutions, and one free variable.

(c) $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 5 \\ 0 & 1 & -2 & 1 & -4 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 2 & 3 & 6 \end{array}\right] \xrightarrow{R4 \to R4 - 2R3} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 5 \\ 0 & 1 & -2 & 1 & -4 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & -1 & -8 \end{array}\right] \leftarrow$ REF

All the variables are in pivot columns, so there are no free variables. The system corresponding to the REF (and so the original system also) has a unique solution, which may be found by back substitution.

$$x_4 = 8$$
$$x_3 = 7 - 2x_4 = -9$$

etc.

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5. Show that the following pair of matrices A and B is **row equivalent**, and find a sequence of elementary row operations (EROs) which converts one into the other.

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$$

Row reduce A and B.

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \xrightarrow{R1 \to R1 - R2} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\xrightarrow{R2 \to \frac{1}{2}R2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R1 \to R1 - R2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ ← RREF of A

$$B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} \xrightarrow{R1 \to R1 - R2} \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \xrightarrow{R2 \to R2 - 3R1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R1 \to R1 - R2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since A and B have the same RREF $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, they are row equivalent. To convert A to B by EROs, convert A to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as above and then $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ to B by performing the <u>opposite</u> of the EROs used to get B to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ <u>in the reverse order</u>:

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \xrightarrow{R1 \to R1 - R2} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\xrightarrow{R2 \to \frac{1}{2}R2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R1 \to R1 - R2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R1 \to R1 + R2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R2 \to R2 + 3R1} \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$$

$$\xrightarrow{R1 \to R1 + R2} \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} = B.$$

6. Consider the following augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & h \\ 1 & 1 & -2 & k \\ 0 & 2 & 1 & h+k \end{array}\right]$$

Find what conditions, if any, must be placed on h and k to ensure that the corresponding system of linear equations is consistent.

Try to reduce to REF:

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & h \\ 1 & 1 & -2 & k \\ 0 & 2 & 1 & h+k \end{array}\right] \xrightarrow{R2 \to R2 - R1} \left[\begin{array}{ccc|c} 1 & -1 & 3 & h \\ 0 & 2 & 1 & k-h \\ 0 & 2 & 1 & h+k \end{array}\right]$$

$$\xrightarrow{R3 \to R3 - R2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & h \\ 0 & 2 & 1 & k-h \\ 0 & 0 & 0 & 2h \end{array}\right] \leftarrow \text{REF.}$$

The system is
$$\begin{cases} x_1 - x_2 + 3x_3 = h \\ 2x_2 + x_3 = k-h \\ 0 = 2h \end{cases}$$

If $h \neq 0$, the 3rd equation has no solution, so the original system is inconsistent.

If $h = 0$, the 3rd equation $0=0$ can be omitted since it always holds and the first two equations can then be solved by back substitution.

So the system is consistent $\iff h = 0$.