Name: SOLUTIONS Date:

Math 20580 Tutorial
Worksheet 10

1. For each of the following, solve the differential equation.

(a) $\frac{dx}{dt} = 2(x^2 + 5)$,

Solution: Seperating the variables

$$\frac{dx}{x^2+5} = 2dt$$

Integrating both sides we get,

$$\begin{aligned} \int \frac{dx}{x^2+5} &= 2t + c \\ \frac{1}{\sqrt{5}} \arctan(\frac{x}{\sqrt{5}}) &= 2t + c \\ x &= \sqrt{5}\tan(2\sqrt{5}t + C) \end{aligned}$$

(b) $x^2 \frac{dy}{dx} = y - 2yx$.

Solution: Seperating variables and integrating we get,

$$\int \frac{dy}{y} = \int \frac{1-2x}{x^2} dx$$

Simplifying the integrals we get,

$$\begin{aligned} \ln y &= -\frac{1}{x} - 2\ln x + c, \\ y &= \frac{Ce^{\frac{-1}{x}}}{x^2}. \end{aligned}$$

Note at $x = 0$ would not be able to solve this and would have to solve based on the intial value problem in different intervals, but the spirit of the problem was to solve the differential equation using variable seperable.

2. Solve the differential equation $y' + 2\sqrt{t}y = \sqrt{t}$. (Hint: Integrating Factor)

Solution: Comparing to the linear differential equation, we first compute the integrating factor,

$$IF = e^{\int 2\sqrt{t}dt} = e^{\frac{4}{3}t^{3/2}}$$

Multiplying the integrating factor across the differential equation, we get

$$\begin{aligned} e^{\frac{4}{3}t^{3/2}}y' + 2e^{\frac{4}{3}t^{3/2}}\sqrt{t}y &= e^{\frac{4}{3}t^{3/2}}\sqrt{t} \\ \frac{d}{dt}\left(e^{\frac{4}{3}t^{3/2}}y\right) &= e^{\frac{4}{3}t^{3/2}}\sqrt{t} \end{aligned}$$

Intergating on both sides we get,

$$e^{\frac{4}{3}t^{3/2}}y = \int e^{\frac{4}{3}t^{3/2}}\sqrt{t}dt.$$

We still need to solve the integral on the right substitute $u = \frac{4}{3}t^{3/2}$ and $du = 2\sqrt{t}$. Now

$$\int e^{\frac{4}{3}t^{3/2}}\sqrt{t}dt = \int \frac{e^u}{2}du = \frac{e^u}{2} + c = \frac{e^{\frac{4}{3}t^{3/2}}}{2} + c$$

Hence,

$$\begin{aligned} e^{\frac{4}{3}t^{3/2}}y &= \frac{e^{\frac{4}{3}t^{3/2}}}{2} + c \\ y &= \frac{1}{2} + \frac{c}{e^{\frac{4}{3}t^{3/2}}} \end{aligned}$$

3. Find the solution to the initial value problem

$$t\frac{dy}{dt} + y = t\sin t, \quad y(\pi) = 2.$$

Then, find the maximal interval of existence of the solution.

Solution: Rewriting as,

$$\frac{dy}{dt} + \frac{y}{t} = \sin t$$

This is again a first order linear duifferential equation and we solve using the integrating factor method. The integrating factor is,

$$e^{\int \frac{dt}{t}} = t$$

Hence note that,

$$\begin{aligned} t\frac{dy}{dx} + y &= t\sin t \\ \frac{d}{dx}(ty) &= t\sin t \end{aligned}$$

Integrating on both sides, we get

$$\begin{aligned} ty &= \sin t - t\cos t + C \\ y &= \frac{\sin t}{t} - \cos t + \frac{C}{t} \end{aligned}$$

Substituting $y(\pi) = 2$,

$$2 = 0 + 1 + \frac{C}{\pi}$$

Solving we get $C = \pi$. Hence,

$$y = \frac{\sin t + \pi}{t} - \cos t.$$

As the function tends to infinity at we approach 0 we cannot solve the IVP beyond zero and hence the maximal interval of existence is of the solution is $(0, \infty)$.

4. Find all the *stable* equilibrium solutions of the autonomous system

$$\frac{dy}{dt} = y^3 - 5y^2 + 6y$$

Compute $\lim_{t\to\infty} y(t)$ where $y(t)$ is the solution of the ODE satisfying the following initial conditions

(a) $y(0) = 0$

(b) $y(0) = 1$

Solution:

$$y^3 - 5y^2 + 6y = y(y - 5y + 6) = y(y - 2)(y - 3)$$

For a stable equilibrium $\frac{dy}{dt}$ changes sign from positive to negative. This happens at $y = 2$.

(a) $y(0) = 0$ will lead to a constant solution. Hence $\lim_{t\to\infty} y(t) = 0$.

(b) $y(0) = 1$: in the interval $(0, 2)$, $\frac{dy}{dt} > 0$ and hence function is increasing and asymptotic to 2 amd $\lim_{t\to\infty} y(t) = 2$.

Refer to the the page 2 of the following notes for a similar example.
We would have three critical points in our example $0, 2$ and 3. Only 2 will be a stable euqilibrium and the other two will be unstable. Any points that start at these equilibrium will remain constant. So for any initial value probelm that starts between $(0, 2)$ will asymptote to 2 as $t \to \infty$.