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Math 20580 (L.A. and D.E.) Tutorial
Worksheet 2

1. Determine whether the vector $\mathbf{w}$ can be written as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. If yes, find scalars a_1, a_2, a_3 such that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{w}$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 5 \\ 3 \\ -6 \end{bmatrix}$$

Recall that to check if $\vec{w}$ is a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$ we can reduce the augmented matrix

$$\begin{array}{cccc} \left[\begin{array}{ccc|c} 1 & 1 & 9 & 5 \\ 0 & 1 & 8 & 3 \\ -2 & 1 & 7 & -6 \end{array}\right] \\ \uparrow\ \ \ \uparrow\ \ \ \uparrow\ \ \ \ \ \uparrow \\ v_1\ \ v_2\ \ v_3\ \ \ w \end{array}$$

Since solving for $a_1 v_1 + a_2 v_2 + a_3 v_3 = w$ is equivalent (by looking entry-by-entry) to solving the system of linear equations

$$\begin{cases} 1\cdot a_1 + 1\cdot a_2 + 9\cdot a_3 = 5 \\ 0\cdot a_1 + 1\cdot a_2 + 8\cdot a_3 = 3 \\ -2\cdot a_1 + 1\cdot a_2 + 7\cdot a_3 = -6 \end{cases}$$

which has that matrix as its augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 9 & 5 \\ 0 & 1 & 8 & 3 \\ -2 & 1 & 7 & -6 \end{array}\right] \underset{}{\overset{R_3+2\cdot R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 9 & 5 \\ 0 & 1 & 8 & 3 \\ 0 & 3 & 25 & 4 \end{array}\right] \overset{R_3-3\cdot R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 9 & 5 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

This is now in row echelon form, and it is consistent, therefore solutions exist for a_1, a_2, a_3 and thus w is a linear combination of v_1, v_2 and v_3. To find a_1, a_2 and a_3 we can either pass to <u>reduced</u> row echelon form or back-substitute:

<u>RREF</u>

$$\overset{R_2-8\cdot R_3}{\underset{R_1-9\cdot R_3}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 50 \\ 0 & 1 & 0 & 43 \\ 0 & 0 & 1 & -5 \end{array}\right] \overset{R_1-R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 43 \\ 0 & 0 & 1 & -5 \end{array}\right] \text{ so } \begin{array}{l} a_1 = 7 \\ a_2 = 43 \\ a_3 = -5 \end{array}$$

wouldve been hard to guess !!

<u>back-substitute</u>:

REF gives us the system

$$\begin{array}{l} a_1 + a_2 + 9a_3 = 5 \xrightarrow[\text{for } a_1]{\text{solve}} a_1 = 5+45-43 = 7 \\ a_2 + 8a_3 = 3 \xrightarrow[\text{for } a_2]{\text{solve}} a_2 = 3+40 = 43 \\ a_3 = -5 \end{array}$$

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2. Find the inverses of the following matrices if it exists

(a) $A = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$ (b) $B = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}$

a) $ad - bc = 4 - (-6) = 10 \neq 0$ so invertible!

Inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Hence

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$$

b) $ad - bc = 3 \times (-3) - (-1) \times 9$

$= -9 - (-9)$

$= 0$

So B is not invertible.

3. (a) Let

$$A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}.$$

Construct a 2×2 matrix B such that AB is the zero matrix. Use two different **nonzero** columns for B.

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $AB = \begin{bmatrix} 6a-3c & 6b-3d \\ -2a+c & -2b+d \end{bmatrix}$

So if $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then comparing entry-by-entry we get that we must

have $\begin{matrix} 6a-3c=0 & 6b-3d=0 \\ -2a+c=0 & -2b+d=0 \end{matrix}$ which implies $c=2a$ and $d=2b$. The question asks for an

example, so choose one! e.g: $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, or $B = \begin{bmatrix} 3 & 9 \\ 6 & 18 \end{bmatrix}$, or $B = \begin{bmatrix} -12 & 5 \\ -24 & 10 \end{bmatrix}$, ...

(b) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and } C = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}.$$

Find the conditions on a, b, c, and d such that A commutes with both B and C, that is, $AB = BA$ and $AC = CA$.

$AB = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$

$AB = BA \iff \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

look entry-by-entry

$BA = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

$\iff \begin{matrix} a=a & b=0 \\ c=0 & 0=0 \end{matrix}$

$\iff b=0$ and $c=0$ $\longrightarrow$ so A could look like $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

$AC = \begin{bmatrix} a-2b & -a+b \\ c-2d & -c+d \end{bmatrix}$

$AC = CA \iff \begin{bmatrix} a-2b & -a+b \\ c-2d & -c+d \end{bmatrix} = \begin{bmatrix} a-c & b-d \\ -2a+c & -2b+d \end{bmatrix}$

$CA = \begin{bmatrix} a-c & b-d \\ -2a+c & -2b+d \end{bmatrix}$

$\iff \begin{matrix} a-2b=a-c & -a+b=b-d \\ c-2d=-2a+c & -c+d=-2b+d \end{matrix}$

$\iff c=2b$ and $a=d$.

Since $b=c=0$ by above we see A must look like $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ for some a.

4. (a) Given the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

find AB, A^{-1} and B^T.

(b) Using your answer from (a) determine $3AB + A^{-1} - 2B^T$.

$$AB = \begin{bmatrix} 1\times 1 + 2\times 0 & 1\times(-2) + 2\times 1 \\ 1\times 1 + 3\times 0 & 1\times(-2) + 3\times 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1\times 3 - 2\times 1} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$3AB + A^{-1} - 2B^T = 3 \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ +4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3+3+(-2) & 0+(-2)+0 \\ 3+(-1)+4 & 3+1+(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 \\ 6 & 2 \end{bmatrix}$$

5. For both of the following vectors, determine if they are in the span of columns of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}.$$

(a) $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

Recall that the span here is the set of all linear combinations of the columns $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. So equivalently we're asking if v is a linear combination of these?

Therefore, just like in Question 1, we can consider the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 3 & 5 \\ 1 & 4 & 8 \end{array}\right] \underset{R_3 - R_1}{\overset{R_2 - R_1}{\sim}} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{array}\right] \overset{R_3 - 2 \cdot R_2}{\sim} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

The corresponding linear system of equations is therefore consistent, so yes it is in the span! (in fact $v = -4\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$)

(b) $\mathbf{w} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$

Similarly to above:

$$\left[\begin{array}{cc|c} 1 & 2 & -4 \\ 1 & 3 & 1 \\ 1 & 4 & 3 \end{array}\right] \underset{R_3 - R_1}{\overset{R_2 - R_1}{\sim}} \left[\begin{array}{cc|c} 1 & 2 & -4 \\ 0 & 1 & 5 \\ 0 & 2 & 7 \end{array}\right] \overset{R_3 - 2R_2}{\sim} \left[\begin{array}{cc|c} 1 & 2 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & -3 \end{array}\right]$$

Last row is equivalent to the equation: $0 = -3$ so the system is inconsistent

Therefore no solutions to $x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$ exist, so $\begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, i.e. it is not in their span.