Name: Date: 01/30/2025

Math 20580 (L.A. and D.E.) Tutorial
Worksheet 3

1. Which of the following transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ are linear?

   (a) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;

   (b) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;

   (c) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$;

   (d) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} |x| \\ |y| \end{bmatrix}$.

Solution:

(a) T is linear since $T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, and $T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = cT\begin{bmatrix} x \\ y \end{bmatrix}$;

(b) T is not linear since $0 \cdot T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix} = T\left(0 \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$;

(c) T is linear since $T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2+y_1+y_2 \\ x_1+x_2-y_1-y_2 \end{bmatrix} = T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, and $T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} cx+cy \\ cx-cy \end{bmatrix} = cT\begin{bmatrix} x \\ y \end{bmatrix}$;

(d) T is not linear since $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 2 \end{bmatrix} = T\begin{bmatrix} 1 \\ 1 \end{bmatrix} + T\begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

2. Which of the following sets of vectors in $\mathbb{R}^3$ are linearly independent?

   (a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$;

   (b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$;

   (c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$.

Solution:

(a) is not linearly independent since $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ has a free variable;

(b) is linearly independent since

$$\left[\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -3 & 0 \\ 0 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

does not have a free variable;

(c) is not linearly independent since

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

has a free variable.

3. Which of the following sets of vectors $\alpha = (a_1, a_2, a_n)$ in $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$?

   (a) The set of all α such that $a_1 \geq 0$;

   (b) The set of all α such that $a_1 + 3a_2 = a_3$;

   (c) The set of all α such that $a_2 = a_1^2$.

Solution:

(a) is not a subspace of $\mathbb{R}^3$ since $(1, 0, 0)$ is in the subset but $(-1)\cdot(1, 0, 0) = (-1, 0, 0)$ is not.

(b) is a subspace of $\mathbb{R}^3$ since for (a_1, a_2, a_3) and (b_1, b_2, b_3) in the subset, we have

$$ca_3 + b_3 = c(a_1 + 3a_2) + (b_1 + 3b_2) = (ca_1 + b_1) + 3(ca_2 + b_2),$$

and thus $c(a_1, a_2, a_3) + (b_1, b_2, b_3)$ is in the subset.

(c) is not a subspace of $\mathbb{R}^3$ since $(1, 1, 0)$ is in the subset but $2 \cdot (1, 1, 0) = (2, 2, 0)$ is not.

4. Given the matrix $A = \begin{bmatrix} 1 & -1 & -2 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 0 & -2 & 10 \end{bmatrix}$, find a basis for Row($A$), Col($A$) and Null($A$).

   Hint: The first step is to find a row echelon form of A. Then, the non-zero rows will form a basis of Row(A), and pivots will indicate which columns of A form a basis of Col(A) (but we do not pick columns of a REF of A for a basis of Col(A)). For Null(A), augment A by zero and solve the resulting system.

Solution: Note that

$$\begin{bmatrix} 1 & -1 & -2 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 0 & -2 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & 3 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & -2 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & 3 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can choose a basis for Row(A) to be $\left\{ \begin{bmatrix} 1 & -1 & -2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 5 \end{bmatrix} \right\}$.
A basis for Col(A) can be chosen as the first and the third columns of A:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

For the null space, augment a row echelon form of A by zero and consider

$$\left[\begin{array}{cccc|c} 1 & -1 & -2 & 3 & 0 \\ 0 & 0 & -1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The free variables are x_2 and x_4, and a basis for Null(A) can be chosen as

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}.$$