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Math 20580 Tutorial
Worksheet 4

1.

(a) Given the matrix $A = \begin{bmatrix} 1 & -1 & -2 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 0 & -2 & 10 \end{bmatrix}$, find a basis for Row($A$).

(b) Find the nullity of A;

(c) Given the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & 5 \\ 4 & 7 \\ 3 & 6 \end{bmatrix}$, find the rank of B.

(d) Find the nullity of B.

$\boxed{(a)}$ $\begin{pmatrix} 1 & -1 & -2 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 0 & -2 & 10 \end{pmatrix} \xrightarrow{R_2+R_1} \begin{pmatrix} 1 & -1 & -2 & 3 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & -2 & 10 \end{pmatrix} \xrightarrow{-R_2} \begin{pmatrix} 1 & -1 & -2 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & -2 & 10 \end{pmatrix}$

$\xrightarrow{R_1+2R_2} \begin{pmatrix} 1 & -1 & 0 & -7 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & -2 & 10 \end{pmatrix} \xrightarrow{R_3+2R_2} \begin{pmatrix} 1 & -1 & 0 & -7 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: A'$.

Since $\text{Row}(A) = \text{Row}(A') := \text{span}\{\text{all nonzero rows of } A'\}$

$= \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \\ -7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 5 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \\ -7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 5 \end{pmatrix}\right\}$ is linearly independent,

$\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \\ -7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 5 \end{pmatrix}\right\}$ is a basis for Row(A).

$\boxed{1(b)}$ Method 1: Since there are 2 free variables in A', nullity of $A = 2$.

Method 2: rank $A = \dim(\text{Row}(A)) = 2$. By rank-nullity Theorem, nullity $A = 4 - \text{rank } A = 4 - 2 = 2$.

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$\boxed{(c)}$ First, rank $B = \dim(\text{Col}(B))$. Since $\left\{\begin{pmatrix}2\\1\\4\\3\end{pmatrix}, \begin{pmatrix}3\\5\\7\\6\end{pmatrix}\right\}$ is linearly independent and $\text{span}\left\{\begin{pmatrix}2\\1\\4\\3\end{pmatrix}, \begin{pmatrix}3\\5\\7\\6\end{pmatrix}\right\} = \text{Col}(B)$, $\left\{\begin{pmatrix}2\\1\\4\\3\end{pmatrix}, \begin{pmatrix}3\\5\\7\\6\end{pmatrix}\right\}$ is a basis for $\text{Col}(B)$ and thus

rank $B = 2$.

$\boxed{(d)}$ By rank-Nullity Thm, nullity $B = 2 - 2 = 0$.

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2. Let $\mathbf{b}_1 = \begin{bmatrix}5\\4\end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix}2\\3\end{bmatrix}$; and $\mathbf{c}_1 = \begin{bmatrix}3\\2\end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix}1\\1\end{bmatrix}$ be two bases for $\mathbb{R}^2$. Find $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

$$\vec{b}_1 = \begin{pmatrix}5\\4\end{pmatrix} = 1\begin{pmatrix}3\\2\end{pmatrix} + 2\begin{pmatrix}1\\1\end{pmatrix} = 1\cdot\vec{c}_1 + 2\cdot\vec{c}_2$$

$$\vec{b}_2 = \begin{pmatrix}2\\3\end{pmatrix} = -1\cdot\begin{pmatrix}3\\2\end{pmatrix} + 5\begin{pmatrix}1\\1\end{pmatrix} = -1\cdot\vec{c}_1 + 5\cdot\vec{c}_2$$

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} | & | \\ [\vec{b}_1]_{\{\vec{c}_1,\vec{c}_2\}} & [\vec{b}_2]_{\{\vec{c}_1,\vec{c}_2\}} \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 5 \end{pmatrix}$$

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(a) Suppose that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ are two bases of $\mathbb{R}^2$. Also suppose that the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is given as

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}.$$

For $\mathbf{v} = 2\mathbf{b}_1 + \mathbf{b}_2$, what is $[\mathbf{v}]_\mathcal{C}$, the $\mathcal{C}$-coordinates for $\mathbf{v}$?

First, $[\vec{v}]_\mathcal{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Since $[v]_\mathcal{C} = P_{\mathcal{C}\leftarrow\mathcal{B}} [\vec{v}]_\mathcal{B}$,

$$[v]_\mathcal{C} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

(b) Find the standard coordinates for $\mathcal{C}$ if $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}$.

The question is to find $P_{E\leftarrow C}$

Note: $P_{E\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}$ and $P_{\mathcal{B}\leftarrow C} = [P_{C\leftarrow\mathcal{B}}]^{-1}$

$$= \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

Therefore, $P_{E\leftarrow C} = P_{E\leftarrow\mathcal{B}} P_{\mathcal{B}\leftarrow C} = \begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

$$= \begin{pmatrix} -1 & 1 \\ -9 & 7 \end{pmatrix}$$

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4. Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\}$ for $\mathbb{R}^3$.

(a) If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, find $\mathbf{x}$ (its coordinate representation in the standard basis).

(b) What is $P_{\mathcal{B}\leftarrow\mathcal{E}}$ where $\mathcal{E}$ is the standard basis?

$\boxed{(a)}$ $[x]_E = P_{E\leftarrow\mathcal{B}}[x]_{\mathcal{B}} = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 4 & 3 \\ 1 & 3 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ -1 \end{pmatrix}$

or by definition, $[x]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ means

$$\vec{x} = 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ -1 \end{pmatrix}.$$

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$\boxed{(b)}$ $P_{\mathcal{B}\leftarrow\mathcal{E}} = \left[P_{\mathcal{E}\leftarrow\mathcal{B}}\right]^{-1} = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 4 & 3 \\ 1 & 3 & 0 \end{pmatrix}^{-1}$

$$= \begin{pmatrix} 9 & -3 & -5 \\ -3 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$$

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