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Math 20580 (L.A. and D.E.) Tutorial
Worksheet 5

1. Determine whether the following sets are subspaces of the respective vector spaces.

   (a) $U = \{A \in \mathrm{M}_{2\times 2}(\mathbb{R}) : A^T = -A\} \subseteq \mathrm{M}_{2\times 2}(\mathbb{R})$ the set of 2×2 antisymmetric matrices with real entries.

   Solution: Yes.

   - $0^T = 0 = -0$, then $0 \in U$.
   - Suppose $A, B \in U$. Then $A^T = -A$ and $B^T = -B$. Therefore,
   $$(A+B)^T = A^T + B^T = (-A) + (-B) = -(A+B).$$
   Thus, $A + B \in U$.
   - Let $c \in \mathbb{R}$ and $A \in U$. We have $A^T = -A$ and so
   $$(cA)^T = c(A^T) = c(-A) = -(cA).$$
   Then, $cA \in U$.

   Hence, U is a subspace of $\mathrm{M}_{2\times 2}(\mathbb{R})$.

   (b) $V = \{p(x) \in \mathcal{P}_2 \;:\; p(1) = 2\} \subseteq \mathcal{P}_2$.

   Solution: No. Observe that the zero polynomial is not in V because
   $$0(1) = 0 \cdot (1)^2 + 0 \cdot 1 + 0 = 0 \neq 2.$$

(c) $W = \{p(x) \in \mathcal{P}_2 \ : \ p(1) = 0\} \subseteq \mathcal{P}_2$.

Solution: Yes.

- The zero polynomial is $0(x) = 0x^2 + 0x + 0$. Observe $0(1) = 0 \cdot (1)^2 + 0 \cdot 1 + 0 = 0$, then $0 \in W$.
- Suppose $p(x), q(x) \in W$. Then $p(1) = 0$ and $q(1) = 0$. We have ,
$$(p+q)(1) = p(1) + q(1) = 0 + 0 = 0.$$
Thus, $(p+q)(x) \in W$.
- Let $c \in \mathbb{R}$ and $p(x) \in W$. We have $p(1) = 0$ and so
$$(cp)(1) = c \cdot p(1) = c \cdot 0 = 0.$$
Then, $(cp)(x) \in W$.

Hence, W is a subspace of $\mathcal{P}_2$.

2. Let $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ be the linear transformation defined in standard coordinates by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$$

a) Find a basis for $\text{Ker}(T)$.

Solution: Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{Ker}(T)$. Then $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0$, that is, $a + d = 0$. Thus, $d = -a$. Therefore,

$$\text{Ker}(T) = \left\{\begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \in \mathbb{R}\right\}$$

Thus, a basis for $\text{Ker}(T)$ is

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$$

b) Is $\text{range}(T) = \mathbb{R}$?

Solution: Yes.

By definition, we know that $\text{range}(T) \subseteq \mathbb{R}$. On the other hand, let k be any real number. Observe that $\begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$ and

$$T\left(\begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}\right) = k + 0 = k$$

then $\mathbb{R} \subseteq \text{range}(T)$.

3. Consider the linear transformation $T :$ defined by

$$T(ax^2 + bx + c) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

a) Find a linear transformation $S : \mathbb{R}^3 \to \mathcal{P}_2$ such that $T \circ S = \text{Id}$ and $S \circ T = \text{Id}$.

Solution: Define S as

$$S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = ax^2 + bx + c.$$

Observe that

$$T \circ S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = T\left(S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)\right) = T(ax^2 + bx + c) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

then $T \circ S = \text{Id}$. Furthermore,

$$S \circ T(ax^2 + bx + c) = S(T(ax^2 + bx + c)) = S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = ax^2 + bx + c$$

so $S \circ T = \text{Id}$.

b) Are T and S isomorphisms?

Solution: Yes, because they are inverses of each other.

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4. Consider the map $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_1(\mathbb{R})$ defined by

$$T(f(x)) = f'(x)$$

and the map $S : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_1(\mathbb{R})$ defined by

$$S(f(x)) = f'(x) + 1.$$

(a) Is T a linear transformation?

Solution: Yes.

Let $c \in \mathbb{R}$, $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$ in $\mathcal{P}_2(\mathbb{R})$. We have

$$\begin{aligned} T(cp(x) + q(x)) &= T((ca_0 + b_0) + (ca_1 + b_1)x + (ca_2 + b_2)x^2) \\ &= (ca_1 + b_1) + 2(ca_2 + b_2)x \\ &= c(a_1 + 2a_2x) + (b_1 + 2b_2x) \\ &= c \cdot p'(x) + q'(x) \\ &= c \cdot T(p(x)) + T(q(x)). \end{aligned}$$

(b) Is S a linear transformation?

Solution: No.

Observe that $S(2 \cdot x) = S(2x) = 2 + 1 = 3$ but $2 \cdot S(x) = 2 \cdot (1 + 1) = 2 \cdot (2) = 4$. Then, $S(2x) \neq 2S(x)$.

(c) Compute the matrix representation of T and S, if they are linear, relative to the standard bases of $\mathcal{P}_2(\mathbb{R})$ and $\mathcal{P}_1(\mathbb{R})$.

Solution: Recall that the standard bases of $\mathcal{P}_2(\mathbb{R})$ and $\mathcal{P}_1(\mathbb{R})$ are $\{1, x, x^2\}$ and $\{1, x\}$ respectively. Since S is a not a linear transformation, we will compute the matrix representation only for T.

Observe that $T(1) = 0$, so $[T(1)] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Moreover, $T(x) = 1$, so $[T(x)] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Finally, $T(x^2) = 2x$, so $[T(x^2)] = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Therefore,

$$[T] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$