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M20580 L.A. and D.E.
Worksheet 6

1. Let $T : \mathcal{P}^2 \to \mathcal{P}^2$ be the linear transformation on the space of degree two polynomials defined by

$$T(p(x)) = p(x+1).$$

Let $\mathcal{E} = \{1, x, x^2\}$ be the standard basis for $\mathcal{P}^2$.

(a) For $q(x) = x^2 + 3x - 5$, compute $T(q(x))$ and $[q(x)]_{\mathcal{E}}$.

(b) Find the matrix of the linear transformation T with respect to the standard basis $\mathcal{E}$, $[T]_{\mathcal{E}\leftarrow\mathcal{E}}$

(c) Compute $[T(q(x))]_{\mathcal{E}} = [T]_{\mathcal{E}\leftarrow\mathcal{E}}[q(x)]_{\mathcal{E}}$ and verify it matches with part (a)

Solution:

(a)

$$\begin{aligned} T(q(x)) &= (x+1)^2 + 3(x+1) - 5 \\ &= x^2 + 2x + 1 + 3x + 3 - 5 \\ &= x^2 + 5x - 1. \end{aligned}$$

and

$$[q(x)]_{\mathcal{E}} = \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}.$$

(b) $T(1) = 1$, $T(x) = x + 1$ and $T(x^2) = x^2 + 2x + 1$, we get

$$[T]_{\mathcal{E}\leftarrow\mathcal{E}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T]_{\mathcal{E}\leftarrow\mathcal{E}}[q(x)]_{\mathcal{E}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}.$$

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2. Compute the determinants

(a) using cofactor expansion (on any convenient row or column)

(i) $\begin{bmatrix} 1 & 8 \\ 5 & 1 \end{bmatrix}$

> ***Solution:*** Using co-factor expression on the first row we get
> $\det = 1 * 1 - 8 * 5 = -39$

(ii) $\begin{bmatrix} 1 & 0 & 3 \\ 5 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

> ***Solution:*** Using co-factor expression on the first row we get
> $\det = 1(1 * 3 - 1 * 2) - 0(5 * 3 - 0 * 2) + 3(5 * 1 - 0 * 1) = 1 - 0 + 15 = 16$

(b) Using the properties of determinants

(i) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

> ***Solution:*** Swapping two rows or columns has the effect of multiplying the determinant by -1. In this example, swaping the second and third column results in the identity matrix which has determinant 1, so the original matrix of part (c), has determinant -1.

(ii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -5 \\ 2 & 2 & 2 \end{bmatrix}$

> ***Solution:*** The third row is a multiple of the first hence determinant is 0.

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3. Use Cramer's rule to solve the linear system

$$\begin{aligned} x + y - z &= 1 \\ x + y + z &= 2 \\ x - y \quad\; &= 5 \end{aligned}$$

Solution: The associated matrix is $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ which has determinant 4 and the associated solution vector is $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

Replacing the first column with the solution vector we obtain the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 5 & -1 & 0 \end{bmatrix}$ which has determinant 13. So $x = 13/4$.

Replacing the second column with the solution vector we obtain the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 5 & 0 \end{bmatrix}$ which has determinant -7. So $y = -7/4$.

Replacing the third column with the solution vector we obtain the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 5 \end{bmatrix}$ which has determinant 2. So $z = 2/4$.

4. Let A and B be $n \times n$ matrices with det $A = 4$ and det $B = -1$. Find the indicated determinants.

   (a) $\det(AB)$

   (b) $\det(A^2)$

   (c) $\det(B^{-1}A)$

   (d) $\det(4A)$

   (e) $\det(2B^T)$

Solution:

(a) $\det(AB) = \det(A)\mathrm{B} = (4)(-1) = -4$

(b) $\det(A^2) = \det(AA) = \det(A)\det(A) = 4^2 = 16$

(c) $\det(B^{-1}A) = \det(B^{-1})\det(A) = \dfrac{1}{\det(B)}\det(A) = \dfrac{1}{-1}(4) = -4.$

(d) $\det(4A) = 4^n\det(A) = 4^{n+1}$

(e) $\det(2B^T) = 2^n\det(B^T) = 2^n\det(B) = -2^n.$